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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO UNI-AXIAL ELASTIC-PLASTIC WAVE INTERACTIONS

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This paper treats the propagation of stress waves through an elastic-plastic medium on the assumption of uni-axial displacement. With the further simplification to a piecewise linear stress-strain curve in terms of engineering stress and strain, wave equations are obtained for the longitudinal stress in both elastic and plastic regions, each with a distinct constant Lagrangian wave speed. The stress distribution in any region is then simply expressed in terms of two wave functions. In a general motion the medium will be divided into a sequence of alternating elastic and plastic regions separated by moving interfaces. A detailed analysis is presented for a single-interface wave interaction under general initial conditions, namely, continuous initial waves in the two directions in both elastic and plastic regions with a non-uniform yield stress in the elastic region.

For different sets of initial conditions six distinct types of solution are shown to exist, and these are classified according to the direction and speed of the interface. In particular, two types involve interface speeds in excess of the elastic wave speed, not, to the authors' knowledge, demonstrated in previous plastic wave treatments, noting the absence of possible shock formation for the present linearized stress-strain laws. Further, it is shown that stress discontinuities cannot form at the interface (or elsewhere) from initially continuous stress profiles.

Associated with the different types of solution are four distinct sets of interface conditions so that there is no common form for the interaction solution. Each of the six types of solution is shown to be consistent with the elastic-plastic model only under a restricted set of initial conditions, and these sets are found to be mutually exclusive for the six types, thus deciding a unique choice for the type of single-interface solution. The six sets, however, are not inclusive of all possible initial conditions, indicating a need for multi-interface solutions in the exceptional situations. Multi-interface solutions may be possible even in the non-exceptional situations, but this possibility is felt to be unlikely.

Finally, it can be noted that the analysis dealing with validity of solution is, for most cases, only local in that it applies in some small neighbourhood of the current point on the interface path, being based on expansions about this point. The results of such local analysis will therefore extend to the case of non-uniform wave speeds arising from non-linear stress-strain laws, provided that no shock is formed in the neighbourhood, but a global solution can no longer be expressed simply in terms of wave functions.

1. Introduction

The propagation of finite amplitude waves through a metal is of importance, for example, in the investigation of dynamic response by controlled tests and in the prediction of explosion effects on structures. Theoretical analysis must be based on constitutive assumptions for the material, and the well established elastic-plastic model (see, for example, Hill (1950)) provides the simplest description which is reasonably realistic. Further approximations, such as rigid-plastic behaviour or incompressibility, allow infinite signal speeds in the material, and so cannot provide an accurate description of the transient effects of dynamic loading. In the present work the elastic-plastic isotropic work-hardening model is taken to describe the constitutive behaviour. This model recognizes two essentially different types of material behaviour, the choice depending on both the current state of stress and loading history of a material element, and in a general motion a given material element may alternate repeatedly between elastic and plastic states. Even simple boundary or initial conditions produce such a situation. In view of this complexity, few exact wave solutions have been obtained, analytically or numerically, and those that have been are restricted to one-dimensional propagation, uni-axial or spherical. A detailed survey and bibliography of dynamic non-elastic behaviour of metals is given in an article by Hopkins (1966).

In an elastic-plastic analysis, a trial solution requires an assumed wave pattern, that is, a prescribed sequence of elastic and plastic regions with respective wave motions, separated by moving interfaces. A suitable choice of pattern may be intuitively clear for simple loading conditions, if not in general, but it is shown in the present investigation that even simple conditions lead to an unforeseen complexity. Having found a suitable wave pattern which leads to a valid solution, that is, stress distributions and histories in each region consistent with the elastic-plastic model, there remains the vital question of uniqueness. Are there other choices of wave pattern which also lead to valid solutions? This is the main theme of the present investigation, which analyzes in detail a wide class of possible wave patterns in uni-axial propagation. It is shown that for a given set of initial conditions the choice of pattern within this class is unique, and, of course, is predetermined by the initial conditions. At the same time, the existence of a valid solution for a wide range of initial conditions is exhibited.

The treatment is based on wave motions in uni-axial displacement, which has the virtue that the nature of the physical arguments is not obscured by analytical detail. This simple displacement geometry is often adopted as an approximation for the wave motion initiated in a plate by normal loading. Further, the wave patterns show much qualitative similarity with the corresponding spherically symmetric system, and although the latter is analytically more involved, certain results have an analogous derivation. A detailed discussion of the equations governing uni-axial dynamic elastic-plastic deformation, for the isotropic work-hardening model, has been given by Morland (1959)—subsequently referred to as [M]. The equations are most simply expressed in terms of engineering stress and strain, and Lagrangian coordinates, and yield wave equations for the longitudinal stress in both elastic and plastic regions. In these variables, with stress and strain measured positive in compression, both elastic and plastic longitudinal stress-strain paths exhibit upward curvature and the corresponding wave speeds are increasing functions of the stress, significant only over a longer plastic path. Compression waves may then form shocks. From the solutions examined in [M], including the formation of a plastic shock, it was concluded that for the typically small curvature of the plastic longitudinal stress-strain path, satisfactory results, qualitative and quantitative, are obtained by assuming uniform, but distinct, elastic and plastic Lagrangian wave speeds. This corresponds to a piece-wise linear stress-strain path, all elastic and plastic paths having respectively the same constant slope. The resulting simplification has considerable analytical advantage, and is adopted in the present investigation since our concern is with principles rather than fine detail.

On this model the longitudinal stress satisfies a wave equation with constant wave speed in both elastic and plastic regions, the elastic wave speed being greater than the plastic wave speed. The stress in any region can in general be expressed simply in terms of two wave functions, representing waves travelling in the two directions.

Although shocks cannot build up according to the linearized theory, stress discontinuities are considered as a convenient mathematical limit for very short waves. By treating them in this spirit and postulating that stress jumps must follow the same stress-strain path as if the stress change occurs continuously, it is shown that stress discontinuities must propagate with one of the two wave speeds. In particular, a jump embracing both an elastic and plastic change separates into the two parts, each travelling with the appropriate wave speed. Thus both continuous and discontinuous disturbances of a given type travel with the same speed, and it is appropriate to refer to elastic and plastic signal speeds.

Any wave pattern thus comprises a sequence of alternating elastic and plastic regions in one space dimension, each in general containing waves moving in the two directions, and separated by interfaces moving along the propagation axis. We define the term wave interaction to mean the motion of adjacent elastic and plastic regions and their common interface. The wave pattern is, in general, composed of a series of wave interactions, each with a distinct interface, which may be treated separately; these we describe as single-interface interactions and solutions. It will be shown, however, that in certain cases no valid single-interface solution exists in the neighbourhood of some critical point in the space-time plane. Then a multi-interface interaction develops, involving two or more distinct interface paths fanning out from the single point. During some subsequent time interval the multi-interface solution requires appropriate simultaneous matching across each of the interfaces involved, and is inherently more intricate than the single-interface situation. No detailed multi-interface solution is obtained in the present work, which is confined to the solution of a general single-interface interaction.

The required matching conditions across a moving interface involve both conservation of mass and momentum, and consistency with the elastic-plastic model on both elastic and plastic sides. The number of unknown quantities to be determined—the wave functions on both sides and the interface velocity—depends on the initial conditions and on the interface speed in comparison with the elastic and plastic wave speeds, together with its direction. Direction here means whether the movement is into the elastic region or into the plastic region. In the former situation, a particle crossing the interface changes from an elastic to a plastic state, with the reverse in the latter situation, and we delineate these two broad classes by the descriptions elastic-plastic interface, and plastic-elastic interface, respectively. Within each class, depending on interface speed, three different cases are theoretically possible for interfaces at which the stress is continuous, and it is shown that in all six cases the elastic-plastic model provides exactly the required number of matching conditions. It is these results which have an analogous extension to the spherically symmetric problem. For a simple case treated in [M] the appropriate conditions were deduced by direct physical arguments, but such deduction fails in general, and has in fact produced incorrect conditions in a treatment of the spherical problem. In each class, one case involves an interface moving with speed in excess of the elastic wave speed, a situation not, to our knowledge, previously considered (noting the absence of shocks in the present linearized theory). Interfaces with stress discontinuities will be discussed below.

It is found that for continuous interfaces there are four distinct sets of conditions applying to the different cases, so that there is no common solution to the interaction for all cases. It is therefore necessary to determine the solution for each case separately, using the appropriate interface conditions. By obtaining all six types of formal solutions for an arbitrary compatible set of initial conditions, the consistency of each one with its assumed interface motion and the elastic-plastic model can be tested. For some conditions it is quickly evident that certain types of solution are not consistent. It is shown that each type of solution is valid only under a particular set of restrictions on the initial conditions and that the six sets of restrictions are mutually exclusive. Thus we demonstrate that only one of the six types of single-interface solutions can be valid for a given set of initial conditions, and furthermore, determine the restrictions which predict the unique choice. At the same time the existence of the valid solution under appropriate conditions is exhibited. However, there are limited sets of initial conditions for which none of the above six solutions is valid, when presumably a multi-interface interaction must develop. The uniqueness and existence proof, then, is limited to single-interface solutions.

Solutions involving stress discontinuities at an interface, omitted from the above discussion, are in fact much simpler in form. Since a discontinuity must propagate with one of the wave speeds, the motion of a discontinous interface is predetermined by the elastic or plastic change it represents and the further interface conditions then give explicit representations for the wave functions in terms of the initial stress wave profiles and initial yield stress distribution. For a continuous interface, the wave function arguments involve the unknown interface path which is itself defined only by an implicit equation. It is easily shown that no valid discontinuous interface solution is possible for initially continuous stress wave profiles, and that no stress discontinuity can be created by a wave interaction. Thus stress discontinuities (in the linearized theory) can occur only through discontinuous boundary loading, and then they propagate in a readily determined manner. Attention here is confined to initially continuous stress wave profiles.

The single interface interaction comprises adjacent elastic and plastic regions separated by a moving interface, and by definition there is some finite neighbourhood of the interface containing no further interface. The most general initial conditions involve stress waves in the two directions in both elastic and plastic regions, and an arbitrary yield stress distribution, consistent with the model, in the elastic region. The latter is only relevant when the interface is in the elastic-plastic class and so moves into the elastic region. After a time, waves from outer regions may reach the interface and provide new initial conditions for a subsequent interaction, but for some finite interval of time the interaction is influenced only by local initial conditions which may be considered in isolation. The initial waves must describe stress changes over the immediately preceding time interval which are consistent with the elastic-plastic model. Any such elastic and plastic wave patterns separated by an interface must represent a valid solution of one of the six types of interaction, perhaps a trivial solution. Further, any of the six types of solution represents a valid set of initial conditions. Thus a treatment of each of the six types of interaction for general initial conditions covers all thirty-six possible cases of successive interactions. That is, one type of interaction followed by another type, including the six trivial cases of successive identical types. The overall solution of an interaction involving repeated changes of the type of solution may therefore be constructed uniquely in a step-by-step manner, provided that single-interface solutions exist at each stage.

Two simple initial value problems are themselves of interest, those involving only single initial waves in both regions. One is the overtaking interaction, necessarily of a plastic wave by an elastic wave in view of their relative speeds, which is treated in [M] but without the uniqueness proof. The second is a meeting interaction between oncoming elastic and plastic waves, which is treated here as the initial interaction. The analysis exhibits the essential features of an interaction solution. It is then straightforward to extend the solutions to include the different types of general initial conditions, and so complete the investigation of all possible situations. The overtaking interaction is a particular case.

2. Governing equations for uni-axial displacement

It is proposed to consider a uni-axial displacement plane wave motion in which all dependent variables are functions only of the time t and a single rectilinear coordinate x. Further, x is chosen to be a Lagrangian coordinate denoting the initial particle position along the axis of motion, or longitudinal direction, as this leads to an exact linear equation of motion. The particle displacement is everywhere directed along the x-axis and is denoted by $u(x, t)$. For this simple displacement field, the deformation is defined by the single non-vanishing principal strain component in the x-direction, and it is most convenient to adopt the engineering measure

$$\epsilon = -\partial u/\partial x, \tag{2·1}$$

which represents the contraction per unit initial length. Defining strain and stress to be positive in compression follows the convention in [M] where the detailed derivation of the governing equations is given; here we will just define notation and present a brief discussion of the main features. The compressive principal stress in the x-direction (with no distinction between initial and current cross section area) is denoted by σ. In addition, there exist non-zero lateral principal stresses which provide the uni-axial displacement constraint, but these are eliminated in the final equations which are written explicitly in terms of the longitudinal stress and strain components σ, ϵ.

In this simple displacement geometry, the isotropic work-hardening model provides an explicit longitudinal stress-strain relation for both elastic and plastic changes of a given material element, identical for both Von Mises and Tresca yield criteria. Elastic changes satisfy the differential relation

$$\frac{d\sigma}{d\epsilon} = \frac{K+\frac{4}{3}\mu}{1-\epsilon}, \tag{2·2}$$

and plastic changes satisfy

$$\frac{d\sigma}{d\epsilon} = \frac{K}{1-\epsilon} + \frac{2}{3(1-\epsilon)}\frac{d\sigma_s}{d\epsilon}. \tag{2·3}$$

In these equations K and μ are respectively the elastic bulk and shear moduli, and $\sigma_s(\epsilon)$ is derived from the plastic stress-strain relation in simple compression. Since the work-hardening contribution, $d\sigma_s/d\epsilon$, is small in comparison with K, it may be neglected so that here the perfectly plastic model provides a good approximation. Upward curvature of the stress-strain paths follows from the denominator $1-\epsilon$, and the assumption that K and μ are increasing functions of pressure and consequently of the compressive strain ϵ. However, the data and wave solution investigated in [M] show that the contribution of this curvature is not important in the overall stress distribution, and accordingly we adopt the simplifying approximations of constant elastic and plastic slopes. The elastic slope is clearly greater than the plastic slope. Integration constants are determined by the stress-strain state at which the change from the elastic to the plastic equation, or vice versa, last occurred.

Typical loading-unloading cycles OYMM′N and OYZZ′N are shown in figure 1. The elastic paths Y′Y, Z′Z, M′M are reversible while the plastic yield path YZM and reverse path M′Z′Y′ are followed only in the sense shown by the arrows. Thus, at a given element x, σ must be non-decreasing on YZM, and non-increasing on M′Z′Y′, both corresponding to a positive rate of plastic working. Y is the initial yield point and Z, M are yield points for an element last plastic at the states Z and M respectively. Hence a region in which different elements have been previously loaded to different levels on the yield path has a corresponding distribution of yield points, that is a non-uniform distribution of longitudinal yield stress. This does not conflict with the approximation of perfect-plasticity as different yield values of the longitudinal stress σ correspond to different points on the fixed yield surface in principal stress space, recalling that there are also non-zero lateral stresses. It is convenient to refer to the yield value of σ as the yield stress, and a varying distribution is described as kinematic hardening. The point N represents the permanent or plastic strain resulting from the complete loading-unloading cycle. In this model the elastic ranges MM′, ZZ′, YY′ are double the initial compression range OY, but this is not essential to the following analysis. All permissible stress-strain states lie between YM and M′Y′ continued.

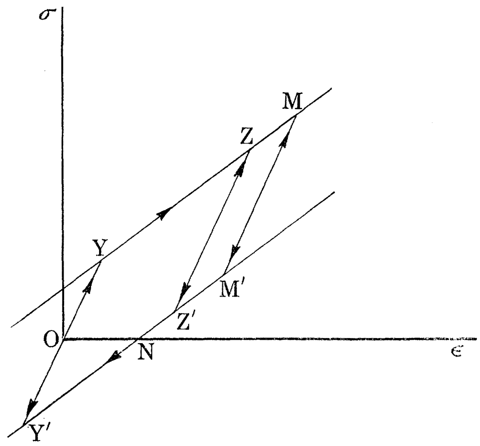

FIGURE 1. Longitudinal stress-strain path.

It remains to make the assertion that in the present rate-independent theory discontinuous changes of stress will be regarded as limits of the corresponding continuous change, and therefore follow (instantaneously) the same stress-strain path. In particular, a jump involving both elastic and plastic changes, as, for example, a jump from state O to the stress level at M, follows the required elastic and plastic paths in succession, in this example OY and YM. That is, the same final state M is attained by the elastic-plastic path OYM prescribed by the model—no alternative paths are permitted.

On the assumption of uniform initial density ρ_0, and neglect of body force, the longitudinal momentum equation is exactly

$$\frac{\partial \sigma}{\partial x} = -\rho_0 \frac{\partial^2 u}{\partial t^2}, \tag{2·4}$$

and the lateral equilibrium equations are already used. The wave pattern separates the axis into a sequence of finite elastic regions and finite plastic regions at each instant of time. In a single plastic region the stress changes at every element follow either the yield path YM or reverse yield path M′Y′, since an exchange of path involves an intermediate elastic region —possibly just an elastic stress discontinuity. Thus in a plastic region there is a common longitudinal stress-strain relation $\sigma = S(\epsilon)$. However, in an elastic region it has been noted that different elements may follow different elastic paths, for example, MM′ or ZZ′ in figure 1, corresponding to different plastic histories; this is illustrated by the overtaking interaction in [M]. Here there is a non-homogeneous stress-strain relation $\sigma = S(\epsilon, x)$, which includes the homogeneous plastic relation as a special case. Thus, differentiating (2·4) partially with respect to x, at constant t, and using (2·1), gives

$$\frac{\partial^2 \epsilon}{\partial t^2} = \frac{1}{\rho_0}\frac{\partial^2 \sigma}{\partial x^2}, \tag{2·5}$$

while differentiating the non-homogeneous stress-strain relation twice with respect to t, at constant x, gives

$$\frac{\partial^2 \sigma}{\partial t^2} = \frac{\partial S}{\partial \epsilon}\frac{\partial^2 \epsilon}{\partial t^2} + \frac{\partial^2 S}{\partial \epsilon^2}\left(\frac{\partial \epsilon}{\partial t}\right)^2. \tag{2·6}$$

Now the slopes of the stress-strain paths are assumed to have the same constant values for all elements so that

$$\partial S/\partial \epsilon = \rho_0 c_\alpha^2 \quad (\alpha = 0, 1), \tag{2·7}$$

where $\alpha = 0, 1$ refer respectively to elastic and plastic paths and c_0, c_1 are constants satisfying

$$c_0 > c_1. \tag{2·8}$$

Thus, in either an elastic or a plastic region $\partial^2 S/\partial \epsilon^2$ vanishes,† and the strain can be eliminated between (2·5) and (2·6), giving

$$\frac{\partial^2 \sigma}{\partial x^2} = \frac{1}{c_\alpha^2}\frac{\partial^2 \sigma}{\partial t^2}. \tag{2·9}$$

That is, the longitudinal stress in elastic and plastic regions satisfies the wave equation with constant Lagrangian wave speeds c_0, c_1 respectively, and the elastic wave speed c_0 is greater than the plastic wave speed c_1.

In both elastic and plastic regions, then, a continuous stress distribution is described in terms of two arbitrary wave functions,

$$\sigma = g(x - c_\alpha t) + h(x + c_\alpha t), \tag{2·10}$$

where g, h represent waves propagating in the positive and negative directions respectively, and depend on initial and boundary conditions.

The other physical variable which explicitly arises in the interface conditions is the particle velocity with respect to fixed axes,

$$v = \partial u/\partial t. \tag{2·11}$$

Along the characteristics of (2·9) the characteristic relations give

$$\sigma \pm \rho_0 c_\alpha v = \text{constant on } \mathrm{d}x/\mathrm{d}t = \pm c_\alpha, \tag{2·12}$$

where, in general, the constant differs between characteristics of both positive and negative families. A useful alternative expression for v in terms of σ is obtained by integrating (2·4) with respect to t at fixed x, namely

$$v = -\frac{1}{\rho_0}\int^t \frac{\partial \sigma}{\partial x}\,\mathrm{d}t. \tag{2·13}$$

† This situation was expressed incorrectly in [M].

Further, v also satisfies the appropriate wave equation in each region, while u and ϵ do so only for the homogeneous stress-strain relation.

It remains to consider the motion of a stress discontinuity, excluded from the above discussion. If the discontinuity surface is propagating instantaneously with velocity V with respect to fixed axes, and conditions on the two sides are denoted by suffixes 1 and 2, conservation of mass and momentum require

$$\rho_2(v_2-V) = \rho_1(v_1-V), \tag{2·14}$$

$$\sigma_2+\rho_2(v_2-V)^2 = \sigma_1+\rho_1(v_1-V)^2. \tag{2·15}$$

Here ρ_1, ρ_2 refer to current densities, and are related to the strain and initial density (mass conservation) by

$$\rho_2(1-\epsilon_2) = \rho_1(1-\epsilon_1) = \rho_0. \tag{2·16}$$

Eliminating V between (2·14) and (2·15), and ρ_1, ρ_2 by (2·16), shows that

$$\rho_0(v_2-v_1)^2 = (\sigma_2-\sigma_1)(\epsilon_2-\epsilon_1). \tag{2·17}$$

If the path of the discontinuity surface in the Lagrangian (x, t) plane is $x = X(t)$, so that its Lagrangian velocity is $\dot{X}(t)$, then since displacement remains continuous across the surface, so does $\partial u/\partial t+\dot{X}\,\partial u/\partial x$, which implies

$$v_2-\dot{X}\epsilon_2 = v_1-\dot{X}\epsilon_1. \tag{2·18}$$

Eliminating v_2-v_1 between (2·17) and (2·18) gives

$$(\dot{X})^2 = \frac{1}{\rho_0}\frac{\sigma_2-\sigma_1}{\epsilon_2-\epsilon_1}. \tag{2·19}$$

Thus, if the stress jump is the limit of a continuous change along a single elastic or single plastic path, with the respective constant slopes defined by (2·7), then

$$(\dot{X})^2 = c_\alpha^2, \quad \epsilon_2-\epsilon_1 = \frac{\sigma_2-\sigma_1}{\rho_0 c_\alpha^2}, \quad v_2-v_1 = \pm\frac{\sigma_2-\sigma_1}{\rho_0 c_\alpha}, \tag{2·20}$$

where the sign of the particle velocity jump is the same as that of $\dot{X}$. That is, elastic and plastic discontinuities propagate (in the Lagrangian coordinate system) with the respective wave speeds. Further, the previous assertion that a jump involving both elastic and plastic changes is the limit of the separate elastic and plastic continuous changes followed in turn, implies that such a discontinuity splits into the corresponding single-state jumps, each of which propagates in the manner described above.

It has been shown that both continuous and discontinuous single-state stress changes propagate, according to the linearized model, along characteristics in the Lagrangian (x, t) plane with the appropriate velocity $\pm c_0$, $\pm c_1$. However, interfaces between elastic and plastic regions, across which stress is continuous but there is a change in the constitutive stress-strain relation, are not confined to propagation along characteristics. In fact, as shown in the next section, such interfaces may propagate with any velocity.

3. Discussion of wave interactions and interface conditions

All practical cases of dynamic loading on a boundary will result in a sequence of alternating elastic and plastic regions separated by moving interfaces. However, such motions in general divide naturally into a series of wave interactions, each involving two adjacent elastic and plastic regions and a single separating interface. Exceptional situations can arise when more than one interface develops at a single point in the (x, t) plane, and then for some subsequent interval of time the motions in all the adjacent regions must be considered simultaneously. These we have termed multi-interface interactions, but since in most situations a valid single-interface solution is exhibited, only the need in particular cases for multi-interface interactions has been indicated, without attempting to analyse essentially more complex solutions. Attention, then, is confined to the single-interface interaction, with, in general, waves in the two directions in both elastic and plastic regions.

The solution of a single-interface interaction involves the determination of all the stress wave functions and the interface path $x = X(t)$. The wave functions are solutions of the equations of motion only within the elastic and plastic regions and it is necessary to satisfy mass and momentum conservation across the interface. If stress is discontinuous at the interface, then by (2·20) the interface velocity $\dot{X}(t)$ has an appropriate value $\pm c_\alpha$ and the particle velocity jump is related to the stress jump. If the stress is continuous at the interface (the case of zero stress jump), then so is the particle velocity, but no restriction is placed on $\dot{X}(t)$. Two conditions are provided in both cases.

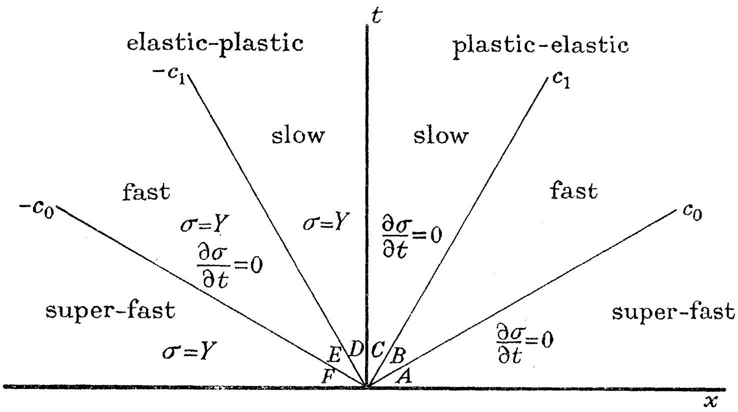

FIGURE 2. Interface conditions for the six ranges of $\dot{X}$. Regions A, C, E open, and B, D, F closed.

Although initial conditions (or equivalent boundary conditions) provide one algebraic relation between the wave functions in each region, their influence domains in the (x, t) plane will not always extend to the interface path on both sides. For example, if an elastic-plastic interface moves faster than a plastic wave, then the initial plastic wave following the interface never overtakes and so does not affect the interaction. In this case a following plastic wave is created and left behind by the interaction, thus introducing a new plastic wave function not directly related to the initial conditions in the plastic region. This is the situation when the interface path lies in region E or F in the (x, t) diagram (figure 2); that is, its velocity is bounded by the respective limits. (The various regions in the figure are defined below—see equation (3·7) *et seq.*) Similarly, for a plastic-elastic interface moving faster than an elastic wave, a path in region A, initial conditions in the elastic region do not directly determine the following elastic wave created by the interaction. In these situations initial conditions provide only one relation between wave functions interacting at the interface. On the other hand, if an elastic-plastic interface moves faster than an elastic wave, a path in region F, there can be no reflected wave on the elastic side, which offsets the loss of an initial condition described above. Similarly, if a plastic-elastic interface moves faster than a plastic wave, a path in region B or A, there can be no reflected wave on the plastic side, while the initial condition is lost only if the path lies in region A.

A summary of the number of relations provided by the conservation of mass and momentum, and initial conditions, and the number of interacting wave functions, is most easily presented by reference to interface paths in each of the regions A to F shown in figure 2. In region B there are four conditions, which are sufficient to determine the three interacting wave functions and the interface path, so no further condition is required or admissable. In regions A and F there are three conditions and three interacting wave functions, so one further condition is needed for a complete solution. In region E there are three conditions and four wave functions, so two further conditions are needed. In regions C and D there are four conditions and four wave functions, so one further condition is needed. The additional conditions follow from the change of state taking place at the interface, which must be consistent with the elastic-plastic model. These are now derived and shown to be precisely the required number in each of the different situations summarized above.

It is supposed that the plastic region is governed by the loading stress-strain path YZM in figure 1, with the current interface state represented by a typical point Z. At a continuous interface the adjacent elastic element must then unload along ZZ′. The alternative case of reverse plastic yielding, M′Z′Y′, and typical elastic reloading, $Z'Z$, is governed by the same analysis if appropriate changes in sign are made to describe the reverse sense of the stress variation. Suffixes e and p are used to denote the evaluation of quantities on the elastic and plastic sides of the interface. At an elastic-plastic interface, the particle immediately on the elastic side is about to become plastic, and the stress there must be at the current yield stress for that particle, that is,

$$\text{(e–p–i)}: \quad \sigma_{\mathrm{e}}[X(t), t] = Y_{\mathrm{e}}[X(t)]. \tag{3·1}$$

$Y(x)$ denotes the yield stress distribution (in the loading or reverse sense as appropriate) which is known in the elastic region from the previous plastic history, and is just the current stress in the plastic region by definition. There is no corresponding condition at a plastic-elastic interface where the particle changes from the plastic state, since the current plastic stress defines the yield stress.

In each case of a stress discontinuity at the interface, the above conditions are precisely the required number. If the interface is elastic-plastic it must propagate with speed c_1, so the four interacting wave functions are determined by the two initial conditions, the particle velocity-stress jump relation, and (3·1). If it is plastic-elastic it must propagate with speed c_0, when there is no reflected wave on the plastic side, so the absence of the additional yield requirement (3·1) is balanced by the loss of one wave function. Some appropriate solutions are investigated in appendix A with regard to the later consideration of validity.

Returning to the case of continuous interfaces it is assumed that initial (or boundary) stress distributions are continuous, including the yield stress distribution $Y(x)$, and further that they are sufficiently differentiable except at a finite number of points where right- and left-hand derivatives exist. The existence of right- and left-hand derivatives allow various conditions to be expressed compactly, but for most of the conclusions it will be seen to be an unnecessary restriction. Stress discontinuities are easily treated separately in view of their simple predetermined motion, and in particular, it is shown that they cannot be created by the interaction of initially continuous waves. The stress at a particle crossing a continuous elastic-plastic interface increases continuously through the current yield value for yielding in the positive sense, that is, the stress does not exceed the yield stress prior to crossing and does not unload immediately afterwards. This implies that necessarily

$$\text{(e–p–i)}: \quad \left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{e}} \geqslant 0, \quad \left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{p}} \geqslant 0, \tag{3·2}$$

with the opposite inequalities for reverse yielding. If the strict equalities hold, the later analysis of validity hinges on higher derivatives, but the above are sufficient for present purposes. The stress at a particle crossing a continuous plastic-elastic interface is previously increasing for positive yielding and then unloads on becoming elastic, so that necessarily

$$\text{(p–e–i)}: \quad \left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{e}} \leqslant 0, \quad \left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{p}} \geqslant 0; \tag{3·3}$$

the inequalities are reversed for reverse yielding. It will now be shown that continuity of stress and particle velocity across the interface imply a dependence between $(\partial\sigma/\partial t)_{\mathrm{e}}$ and $(\partial\sigma/\partial t)_{\mathrm{p}}$, and compatibility of the inequalities in (3·2) and (3·3) lead to the additional conditions. The analysis is the same as that given by Lee (1953) for interfaces in a simple tension system, but is presented for completeness.

Continuity of σ and v across the interface implies

$$\left.\begin{aligned} \left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{e}} + \dot{X}\left(\frac{\partial \sigma}{\partial x}\right)_{\mathrm{e}} &= \left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{p}} + \dot{X}\left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{p}}, \\ \left(\frac{\partial v}{\partial t}\right)_{\mathrm{e}} + \dot{X}\left(\frac{\partial v}{\partial x}\right)_{\mathrm{e}} &= \left(\frac{\partial v}{\partial t}\right)_{\mathrm{p}} + \dot{X}\left(\frac{\partial v}{\partial x}\right)_{\mathrm{p}}. \end{aligned}\right\} \tag{3·4}$$

Using the equation of motion (2·4) to relate $\partial v/\partial t$ and $\partial\sigma/\partial x$ on each side, then combining the above to eliminate $\partial\sigma/\partial x$, shows that

$$\left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{e}} + \rho_0 \dot{X}^2\left(\frac{\partial v}{\partial x}\right)_{\mathrm{e}} = \left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{p}} + \rho_0 \dot{X}^2\left(\frac{\partial v}{\partial x}\right)_{\mathrm{p}}. \tag{3·5}$$

By the definitions of strain and particle velocity, (2·1) and (2·11), $\partial v/\partial x$ can be replaced by $-\partial\epsilon/\partial t$, which in turn is related to $\partial\sigma/\partial t$ on each side by the respective stress-strain relations and wave speed definitions (2·7), so that (3·5) finally becomes

$$\left(1-\frac{\dot{X}^2}{c_0^2}\right)\left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{e}} = \left(1-\frac{\dot{X}^2}{c_1^2}\right)\left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{p}}. \tag{3·6}$$

(A similar result is obtained in the spherically symmetric system, together with the subsequent conclusions, and will be published separately.)

Compatibility of the inequality pairs (3·2) and (3·3) with (3·6) is now seen to depend on the magnitude of $\dot{X}$ in comparison with c_0 and c_1, that is, on the signs of the two factors containing $\dot{X}$, and in certain cases the only possibility is that the equality signs hold in (3·2), (3·3). This suggests a natural division into six types of interface, three in each class, which are listed below with descriptive titles and labels, and summarized for easy reference in figure 2. It is convenient to make the elastic region $x < X(t)$, and the plastic region $x > X(t)$, so that the elastic-plastic and plastic-elastic classes are distinguished simply by sign of $\dot{X}$.

The complete sets of interface conditions for each type, as given by (3·1) to (3·3), (3·6), are as follows:

Type A: $\dot{X} > c_0$, super-fast plastic-elastic interface

$$\left(\frac{\partial\sigma}{\partial t}\right)_e = \left(\frac{\partial\sigma}{\partial t}\right)_p = 0. \tag{3·7}$$

Type B: $c_0 \geqslant \dot{X} \geqslant c_1$, fast plastic-elastic interface

$$\left(\frac{\partial\sigma}{\partial t}\right)_e \leqslant 0, \quad \left(\frac{\partial\sigma}{\partial t}\right)_p \geqslant 0. \tag{3·8}$$

Type C: $c_1 > \dot{X} > 0$, slow plastic-elastic interface

$$\left(\frac{\partial\sigma}{\partial t}\right)_e = \left(\frac{\partial\sigma}{\partial t}\right)_p = 0. \tag{3·9}$$

Type D: $0 \geqslant \dot{X} \geqslant -c_1$, slow elastic-plastic interface

$$\sigma_e = Y, \quad \left(\frac{\partial\sigma}{\partial t}\right)_e \geqslant 0, \quad \left(\frac{\partial\sigma}{\partial t}\right)_p \geqslant 0. \tag{3·10}$$

Type E: $-c_1 > \dot{X} > -c_0$, fast elastic-plastic interface

$$\sigma_e = Y, \quad \left(\frac{\partial\sigma}{\partial t}\right)_e = \left(\frac{\partial\sigma}{\partial t}\right)_p = 0. \tag{3·11}$$

Type F: $-c_0 \geqslant \dot{X}$, super-fast elastic-plastic interface

$$\sigma_e = Y, \quad \left(\frac{\partial\sigma}{\partial t}\right)_e \geqslant 0, \quad \left(\frac{\partial\sigma}{\partial t}\right)_p \geqslant 0. \tag{3·12}$$

The inclusion of the end-points of the velocity ranges, $\dot{X} = 0$, $\pm c_1$, $\pm c_0$, in particular categories is chosen so that the stated conditions have no exceptions.

The additional interface condition provided by the last continuity argument is the vanishing of $(\partial\sigma/\partial t)_p$ in types A, C, and E. The simultaneous vanishing of $(\partial\sigma/\partial t)_e$ follows automatically and is not a further condition. In the present geometry this is equivalent to the condition that the rate of plastic work vanishes at the interface, and holds only at two of the three types of plastic-elastic interface, and also at one type of elastic-plastic interface. This condition has been postulated by Hopkins (1960), in the case of spherical wave propagation, for all (in our terminology), plastic-elastic interfaces, irrespective of speed. As mentioned earlier, the above analysis extends to the spherical problem, and leads to the same division of interface conditions into six types and the corresponding conclusions for the rate of plastic work vanishing. The inequalities deduced for types B, D, and F are part of the subsequent test of validity, and do not serve in the determination of the solution. Thus the elastic-plastic model requires no further restriction for type B, provides one condition for types A, C, D, and F, and two conditions for type E. These are precisely the requirements in each of the different situations summarized earlier.

Two of the types of interface described above, A and F, move faster than an elastic wave, while it has been shown that all stress disturbances, or signals, propagate with either the plastic or elastic wave speed. The occurrence of a super-fast interface can be shown by a simple example, which both illustrates the mechanism and makes clear the lack of conflict. Consider a simple elastic loading wave travelling through a region of uniform yield stress, necessarily exceeding the maximum stress carried by the wave, but approaching a region in which the yield stress decreases continuously in the direction of propagation, falling to a level below the maximum stress in the wave. Figure 3*a* depicts the wave and yield stress profiles at an instant t_0 when the wave first raises the stress to the yield level, at some particle x_0. As the wave proceeds, particles ahead of x_0 are successively raised to their respective

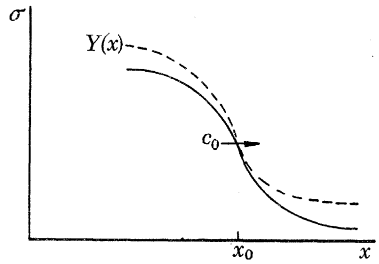

FIGURE 3 (*a*). Initial elastic stress wave profile —, and yield stress distribution ---.

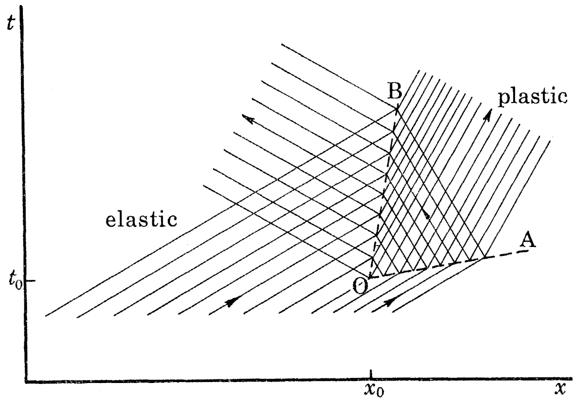

FIGURE 3 (*b*). Characteristics diagram.

yield stress and continue to be loaded, so that an elastic-plastic interface travels forward from x_0. Its path OA, drawn straight for clarity, is shown in a schematic characteristics diagram (figure 3*b*) which includes also a second interface path OB in view of the elastic region behind the wave—a multi-interface interaction. Now the slope of OA is governed by the time delay beyond t_0 for each particle x, $> x_0$, to reach its yield stress, which may occur in as short a time as desired if the yield stress gradient is sufficiently great in comparison with the wave profile gradient. Thus the slope of OA may take any positive value, implying no restriction on the interface speed. In the limit, if the wave and yield stress profiles coincide in some finite region, yield occurs instantaneously over this region so that OA has zero slope, corresponding to an infinite interface speed. The super-fast interface here is the direct consequence of the particular non-uniform yield stress, a material non-homogeneity arising from previous plastic deformation, which causes the change of state at each particle independently; no stress disturbances are required to signal the change of state to successive interface particles.

As an interaction proceeds, the waves currently incident on the interface stem from increasingly distant parts of the initial wave profiles, and in general the type of interaction, as defined above, will be changing. Furthermore, the resulting wave patterns outside the neighbourhood of the interface may themselves lead to changes of state and initiate new interactions. If no multi-interface interaction develops, a given interaction may be treated in isolation from the others in the sense that a solution may be obtained, and tested for validity, in some finite neighbourhood of a chosen initial point on the interface path. In this way the interaction may be followed step by step, but conditions for the existence of a unique solution have, in general, only local validity. The analytic solutions for the interface path and stress wave profiles are given by implicit algebraic relations in terms of the initial wave functions. An analysis of validity relies on expansions about the current interface point. In principle such local expansions should extend to the non-uniform wave velocity case, provided that no shocks form in the neighbourhood.

It will become clear as solutions for each of the six types of interface are obtained, that they cover all possible patterns of waves interacting with a single interface. This is most easily seen from the characteristics diagrams. The only restriction that may be placed on the initial stress wave profiles in both regions, and on the yield stress distribution in the elastic region, is that they are consistent with the elastic-plastic model. The last two observations indicate that any valid pattern of initial waves itself represent a solution of one of the six types of interaction, and further that any such solution represents a valid set of initial conditions. Thus any change of type of solution which arises in an overall interaction is covered by the analysis of the general initial value problem.

The initial meeting interaction between oncoming plastic (here loading) and elastic waves—a case of single initial waves in both regions—is of basic interest, and we choose to treat this first in §§4 and 5. Construction of formal solutions and testing for their validity involves fewer functions than in the general interaction, though the methods are the same, and the essential features are perhaps more easily seen. It is shown that a valid initial solution can break down after some time interval, and the continuation of the interaction again requires formal consideration of the six types of solution under the new initial conditions. For even quite simple initial conditions, an interaction solution must be constructed step by step in this way.

Solutions for the different types of general initial conditions are obtained in §6 by straightforward extensions of those for the meeting interaction. The overtaking interaction, which also involves only single elastic and plastic incident waves, is a particular case.

4. Initial interaction of two meeting waves

The interaction is supposed to start at $t = 0$, with the two oncoming wave fronts making first contact at $x = 0$. Initially there is a single wave travelling in the positive x-direction in the elastic region $x < 0$, and a single wave travelling in the negative direction in the plastic region $x > 0$. Thus, prior to the interaction, the stress distribution is described by

$$t \leqslant 0: \quad \sigma(x,t) = \left.\begin{matrix} G(x-c_0 t) & (x \leqslant c_0 t), \\ G(0) & (c_0 t \leqslant x \leqslant 0), \end{matrix}\right\} \tag{4·1}$$

$$t \leqslant 0: \quad \sigma(x,t) = \left.\begin{matrix} F(x+c_1 t) & (x \geqslant -c_1 t), \\ F(0) & (0 \leqslant x \leqslant -c_1 t). \end{matrix}\right\} \tag{4·2}$$

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Here the initial wave functions G, F, defined in $x \leqslant 0$, $x \geqslant 0$ respectively, are continuous and bounded, and for $t < 0$ the stress in the region between the oncoming waves has the uniform value $G(0) = F(0)$. At $t = 0$, there is a continuous yield stress distribution $Y(x)$ in the elastic region $x < 0$, and since $x = 0$ is the front of the oncoming plastic wave and hence just at yield,

$$Y(0) = F(0) = G(0). \tag{4·3}$$

Without loss of generality it is assumed that the plastic wave is loading, following some section of the yield path YZM in figure 1, and then the elastic wave carries stresses below the yield level; it is not necessary that the initial elastic wave is strictly unloading at the front. Thus

$$\left.\begin{aligned} F'(x) &\geqslant 0 \qquad (x \geqslant 0);\\ G(x) &\leqslant Y(x) \quad (x \leqslant 0);\end{aligned}\right\} \tag{4·4}$$

where the prime denotes differentiation with respect to argument, and appropriate right- and left-hand derivatives are assumed to exist. Further, since the region $x < 0$ has, during $t < 0$, been subjected to an elastic wave carrying a stress $Y(0)$,

$$Y(x) \geqslant Y(0) \quad (x \leqslant 0), \tag{4·5}$$

which is the only explicit limitation that may be placed on $Y(x)$. It is assumed for simplicity that the initial stress profiles extend to infinity, since more remote disturbances of a different nature require a finite time to reach the interaction region and influence the solution. Once this occurs the situation is treated as a new interaction. The initial stress wave profiles are shown in figure 4, there with the elastic wave totally unloading.

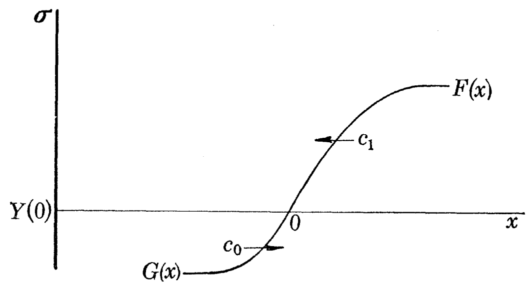

FIGURE 4. Initial stress wave profiles for meeting interaction.

Once the interaction has started there will, in general, be waves in the two directions in both elastic and plastic regions, which are separated by an interface at $x = X(t)$. Thus the general stress distribution for $t \geqslant 0$ is described by

$$\sigma(x,t) = g(x-c_0 t)+h(x+c_0 t) \quad (x < X(t)), \tag{4·6}$$

$$\sigma(x,t) = e(x-c_1 t)+f(x+c_1 t) \quad (x > X(t)). \tag{4·7}$$

In the particular situations when only a single wave exists in one of the regions, the unrequired wave function is set to zero or a constant as appropriate. The above notation for waves in the respective directions in the elastic and plastic regions will be adopted consistently throughout the text. Thus G, g represent waves travelling in the positive direction in

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the elastic region, and H, h waves in the negative direction in the elastic region. Similarly E, e for the positive direction, and F, f for the negative direction, in the plastic region. The significance of each term is then immediately apparent, and subsequently allows omission of arguments without ambiguity, which results in considerably more compact expressions. Furthermore, prescribed initial wave functions, and yield stress distribution, are denoted by capitals, while the lower case symbols denote wave functions to be determined in the interaction solution. The exception to this rule is the interface path, $x = X(t)$, which is also part of the solution, but this should not be misleading.

Formal solutions to the initial-value problem defined by (4·1) to (4·5) may be obtained for each of the six possible types of interface, A to F, described in § 3. It is shown in appendix A that, for general continuous initial stress distributions, no valid solution exists in which a stress discontinuity develops at the interface, or elsewhere. That is, stress discontinuities can arise only by prescription in the initial (boundary) conditions. Thus only the six types of continuous interface need be considered.† Each solution starts with an assumed range for the interface velocity, $\dot{X}(t)$, and the application of interface conditions appropriate to that type. Hence a valid solution must, in the first place, lead to an interface velocity lying at all times within the assumed range. Secondly, the stress distributions in the elastic and plastic regions must be consistent with the elastic-plastic model. That is, stress must not exceed yield anywhere in the elastic region, and must nowhere unload in the plastic region. These aspects of the solutions are deferred until § 5, where existence and uniqueness are discussed. It may be noted that, in view of the implicit nature of the solutions, explicit conclusions are possible only (in general) within some finite distance of the interface and during some finite time interval. In the following five subsections, solutions of types C, B, D, E and F are derived, anticipating that type A cannot be initiated by the present initial meeting waves. This is shown in § 6, where solutions to a general interaction are derived, and in particular it is seen that a valid solution of type A requires more general initial conditions.

4·1. *Solution for a slow plastic-elastic interface, type C*

In this solution the interface velocity lies in the range $0 < \dot{X} < c_1$, so that both initial waves reach the interface and both reflected waves exist. This case is illustrated by the characteristics diagram (figure 5) in which (and in subsequent diagrams) the interface path, OP, is shown as a straight (dashed) line for clarity. The reflected elastic wave $h(x+c_0t)$ exists only in $x \geqslant -c_0t$, but in order to use the representation (4·6) for all $x < X(t)$, $t \geqslant 0$, it is convenient to set $h(x) \equiv 0$ for $x \leqslant 0$, and similarly set $e(x) \equiv 0$ for $x \geqslant 0$. Applying the initial conditions (4·1), (4·2), at $t = 0$, to the representations (4·6), (4·7), shows that

$$g(x) = G(x) \quad (x \leqslant 0); \quad f(x) = F(x) \quad (x \geqslant 0). \tag{4·1·1}$$

Thus the stress distribution for $t \geqslant 0$ becomes

$$\sigma(x,t) = G(x-c_0t)+h(x+c_0t) \quad (x < X(t)), \tag{4·1·2}$$

$$\sigma(x,t) = e(x-c_1t)+F(x+c_1t) \quad (x > X(t)). \tag{4·1·3}$$

† A solution of type B to a meeting interaction is given in [M], but validity and alternative solutions are not discussed.

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Continuity of stress at the interface $x = X(t)$ now gives

$$t \geqslant 0: \quad G[X(t)-c_0 t]+h[X(t)+c_0 t] = e[X(t)-c_1 t]+F[X(t)+c_1 t]. \tag{4·1·4}$$

In order to apply continuity of particle velocity at the interface, expressions for v must be obtained at a generic point P on the interface path (figure 5), is approached from both elastic and plastic sides. In view of the continuity both limits may be denoted by v_P. Further, quantities evaluated at labelled points in the (x, t) diagram will be denoted by the appropriate capital suffix, and in particular, for wave functions, the respective argument is

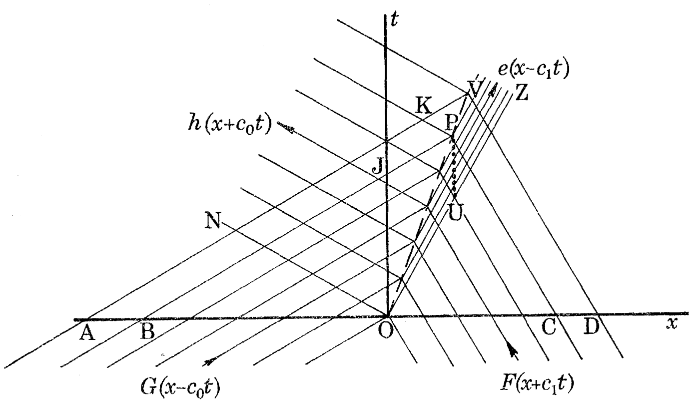

FIGURE 5. Characteristics diagram for meeting interaction with slow plastic-elastic interface, type C; interface path shown ---.

evaluated at the point. Thus G_P is written for $G(x_P - c_0 t_P)$, etc. Applying the characteristic relation (2·12) along the positive characteristic JP in the elastic region, on which the wave function G is constant, shows that

$$\rho_0 c_0(v_P - v_J) = h_J - h_P, \tag{4·1·5}$$

while evaluating (2·13) along the particle path OJ gives

$$\rho_0 c_0(v_J - v_O) = G_J - G_O - h_J + h_O. \tag{4·1·6}$$

Adding (4·1·5) and (4·1·6), and again using $G_J = G_P$, gives the elastic relation†

$$\rho_0 c_0(v_P - v_O) = G_P - G_O - h_P + h_O. \tag{4·1·7}$$

Similarly, following the characteristic OU, along which e is constant, and the particle path UP in the plastic region, gives also

$$\rho_0 c_1(v_P - v_O) = e_P - e_O - F_P + F_O. \tag{4·1·8}$$

Eliminating v_O between (4·1·7) and (4·1·8), and recalling (4·3), shows that

$$c_1(G_P - Y_O - h_P) = c_0(e_P + Y_O - F_P), \tag{4·1·9}$$

where Y_O, $= Y(0)$, is the uniform stress between the initial oncoming waves.

† The derivation in [M] was based solely on a particle path integration to P, and ignored the fact that this path lies in the plastic region; the correct result is obtained.

Combining (4·1·4) and (4·1·9) gives for $t \geqslant 0$,

$$h[X(t)+c_0 t] = \frac{c_0-c_1}{c_0+c_1}\{Y_0 - G[X(t)-c_0 t]\} + \frac{2c_0}{c_0+c_1}\{F[X(t)+c_1 t]-Y_0\}, \tag{4·1·10}$$

$$e[X(t)-c_1 t] = \frac{-2c_1}{c_0+c_1}\{Y_0 - G[X(t)-c_0 t]\} + \frac{c_0-c_1}{c_0+c_1}\{F[X(t)+c_1 t]-Y_0\}. \tag{4·1·11}$$

Since $0 \leqslant X(t) < c_1 t$ for $t > 0$, these are expressions for $h(x)$ in $x \geqslant 0$, and $e(x)$ in $x \leqslant 0$, once $X(t)$ is determined. More specifically, they define the wave functions h and e within the respective domains of the characteristics 'reflected' from the interface up to the current time t. That is, if P is the current point on the interface path, up to the limiting characteristics PK and PV respectively, in figure 5.

It remains to satisfy the change of state conditions at the interface appropriate to type C, namely (3·9). There is just the single condition, the vanishing of $\partial\sigma/\partial t$ on either side, which is sufficient to determine $X(t)$ and complete the solution. Differentiating (4·1·3) and applying (3·9) on the plastic side, shows that for $t \geqslant 0$,

$$e'[X(t)-c_1 t] - F'[X(t)+c_1 t] = 0. \tag{4·1·12}$$

Now differentiating (4·1·11) and eliminating e', noting the strict inequalities $\dot{X} < c_1 < c_0$, it follows that for $t \geqslant 0$,

$$G'[X(t)-c_0 t] = F'[X(t)+c_1 t]. \tag{4·1·13}$$

This is an implicit algebraic equation for $X(t)$ in terms of the given initial functions $G(x)$, $F(x)$. Since $X(0) = 0$, (4·1·13) requires that $G'(0) = F'(0)$, which is a severe restriction, so that solutions of type C are not likely to occur very often. Differentiating (4·1·13) gives an expression for the interface velocity,

$$\{G''[X(t)-c_0 t] - F''[X(t)+c_1 t]\}\dot{X}(t) = c_0 G''[X(t)-c_0 t] + c_1 F''[X(t)+c_1 t], \tag{4·1·14}$$

which is also a first-order differential equation for $X(t)$, with an initial condition $X(0) = 0$.

Conditions for the validity of this solution are determined in § 5·1.

4·2. *Solution for a fast plastic-elastic interface, type B*

Here the interface velocity lies in the range $c_1 \leqslant \dot{X} \leqslant c_0$, and the corresponding characteristics diagram is shown in figure 6. Since $\dot{X} \geqslant c_1$, there are no reflected plastic waves (or positive characteristics leaving the interface path in the plastic region), which means that $e(x-c_1 t) \equiv 0$ for $x \leqslant c_1 t$, and hence everywhere, in the representation (4·7). The situation when the interface overtakes such plastic waves travelling in the positive direction is treated in § 6. Now the initial conditions (4·1·1) and the stress continuity relation (4·1·4) and particle velocity continuity relation (4·1·9), derived for type C, are again obtained with $e(x) \equiv 0$. Thus the resulting expression for $h[X(t)+c_0 t]$ is again (4·1·10), while (4·1·11) becomes an implicit equation for $X(t)$ in $t \geqslant 0$,

$$2c_1\{Y_0 - G[X(t)-c_0 t]\} = (c_0-c_1)\{F[X(t)+c_1 t]-Y_0\}. \tag{4·2·1}$$

Using (4·2·1) allows (4·1·10) to be expressed in the alternative forms

$$h[X(t)+c_0 t] = \frac{c_0+c_1}{c_0-c_1}\{Y_0 - G[X(t)-c_0 t]\},$$

$$= \frac{c_0+c_1}{2c_1}\{F[X(t)+c_1 t]-Y_0\}, \tag{4·2·2}$$

either of which determines $h(x+c_0 t)$ up to the limiting reflected characteristic PK in figure 6, if P is the current point on the interface path. A complete solution is now determined, consistent with the absence of further change of state conditions shown by (3·8).

The interface path equation (4·2·1) is trivally satisfied at $t = 0$, recalling (4·3) and $X(0) = 0$. Differentiating gives an expression for the interface velocity,

$$\{2c_1 G'[X(t)-c_0 t] + (c_0-c_1) F'[X(t)+c_1 t]\}\dot{X}(t) = c_1\{2c_0 G'[X(t)-c_0 t] - (c_0-c_1) F'[X(t)+c_1 t]\}. \quad (4·2·3)$$

Conditions for the validity of this solution are determined in §5·2.

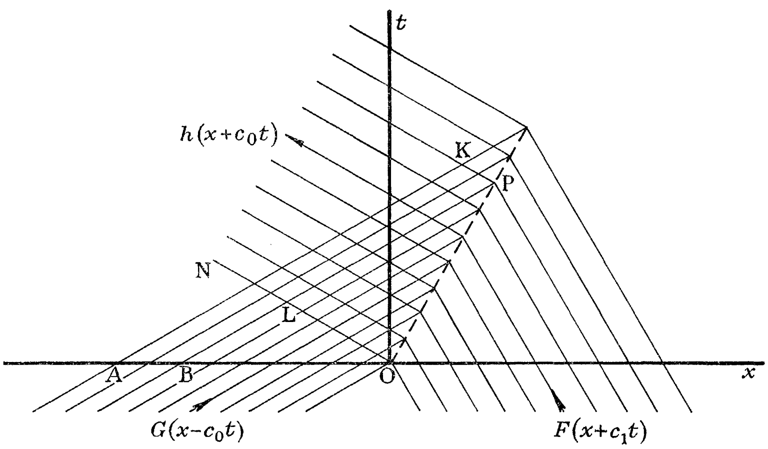

FIGURE 6. Characteristics diagram for meeting interaction with fast plastic-elastic interface, type B; interface path shown ---.

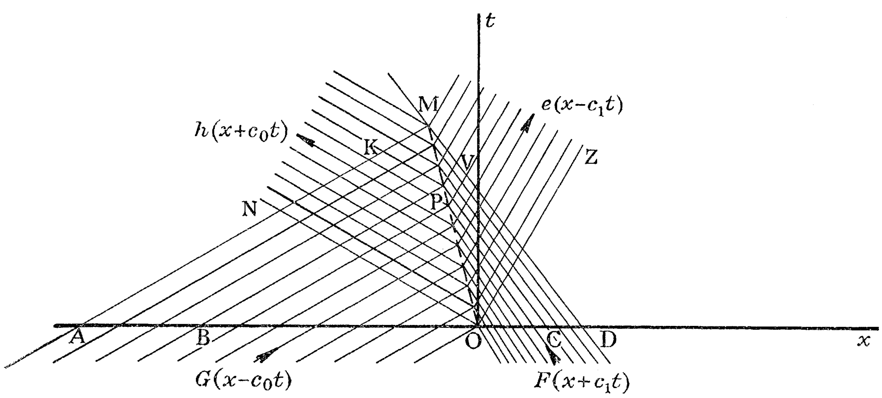

FIGURE 7. Characteristics diagram for meeting interaction with slow elastic-plastic interface, type D; interface path shown ---.

4·3. *Solution for a slow elastic-plastic interface, type D*

Here the interface velocity lies in the range $0 \geqslant \dot{X} \geqslant -c_1$, and the corresponding characteristics diagram is shown in figure 7. As for type C, both reflected waves, $h(x+c_0 t)$, $e(x-c_1 t)$, are present, and the initial and continuity conditions lead again to the expressions (4·1·10) and (4·1·11). Now the single change of state condition, given by (3·10), is that the interface stress is equal to the yield stress at the particle. Thus, using the elastic representation (4·1·2), it follows that

$$G[X(t)-c_0t]+h[X(t)+c_0t] = Y[X(t)], \tag{4·3·1}$$

and substituting for h from (4·1·10),

$$2c_0\{F[X(t)+c_1t]-Y_0\}-2c_1\{Y_0-G[X(t)-c_0t]\} = (c_0+c_1)\{Y[X(t)]-Y_0\}. \tag{4·3·2}$$

This implicit equation for $X(t)$, trivially satisfied at $t = 0$, completes the solution. Differentiating gives an expression for the interface velocity,

$$\{2c_0F'[X(t)+c_1t]+2c_1G'[X(t)-c_0t]-(c_0+c_1)\,Y'[X(t)]\}\dot{X}(t)$$
$$= -2c_0c_1\{F'[X(t)+c_1t]-G'[X(t)-c_0t]\}. \tag{4·3·3}$$

Conditions for the validity of this solution are determined in §5·3.

4·4. *Solution for a fast elastic-plastic interface, type E*

Here the interface velocity lies in the range $-c_1 > \dot{X} > -c_0$, and the corresponding characteristics diagram is shown in figure 8. It is immediately clear that the initial plastic wave $F(x+c_1t)$ is not incident on the interface and that a new plastic wave $f(x+c_1t)$ is continuously created by the interaction and follows behind the interface at slower speed.

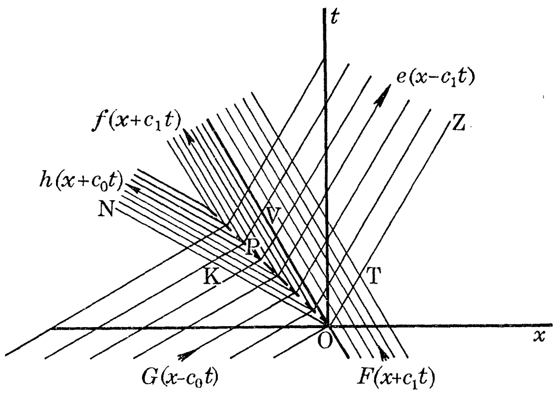

FIGURE 8. Characteristics diagram for meeting interaction with fast elastic-plastic interface, type E; interface path shown ---.

There is a reflected elastic wave $h(x+c_0t)$ and reflected plastic wave $e(x-c_1t)$. Again initial and continuity conditions lead to the expressions (4·1·10) and (4·1·11), here with F replaced by the unknown wave function f. Two further equations are given by the interface conditions (3·11), which require that

$$G[X(t)-c_0t]+h[X(t)+c_0t] = Y[X(t)], \tag{4·4·1}$$

$$e'[X(t)-c_1t]-f'[X(t)+c_1t] = 0. \tag{4·4·2}$$

Now differentiating (4·1·10) and (4·1·11) and using (4·4·2) shows that

$$f'[X(t)+c_1t] = e'[X(t)-c_1t] = h'[X(t)+c_0t] = G'[X(t)-c_0t], \tag{4·4·3}$$

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while differentiating (4·4·1) and eliminating h' by (4·4·3) implies that

$$2G'[X(t)-c_0 t] = Y'[X(t)]. \tag{4·4·4}$$

This is an implicit equation for $X(t)$, and can be satisfied initially only if $2G'(0) = Y'(0)$, which restriction will make solutions of type E uncommon. Differentiating (4·4·4) gives an expression for the interface velocity,

$$\{2G''[X(t)-c_0 t]-Y''[X(t)]\}\dot{X}(t) = 2c_0 G''[X(t)-c_0 t]. \tag{4·4·5}$$

Conditions for the validity of this solution are determined in §5·4.

4·5. *Solution for a super-fast elastic-plastic interface, type F*

Here the interface velocity lies in the range $\dot{X} \leqslant -c_0$, and the corresponding characteristics diagram is shown in figure 9. Again the initial plastic wave $F(x+c_1 t)$ is not incident on the interface and the new following wave $f(x+c_1 t)$ is created, but now the reflected elastic wave $h(x+c_1 t)$ is absent. Thus (4·1·10) and (4·1·11) apply with $h \equiv 0$ and F replaced by f, and the further interface condition (3·12) gives the implicit equation for $X(t)$,

$$G[X(t)-c_0 t] = Y[X(t)], \tag{4·5·1}$$

trivially satisfied at O. Differentiating gives the interface velocity expression

$$\{G'[X(t)-c_0 t]-Y'[X(t)]\}\dot{X}(t) = c_0 G'[X(t)-c_0 t]. \tag{4·5·2}$$

Conditions for the validity of this solution are determined in §5·5.

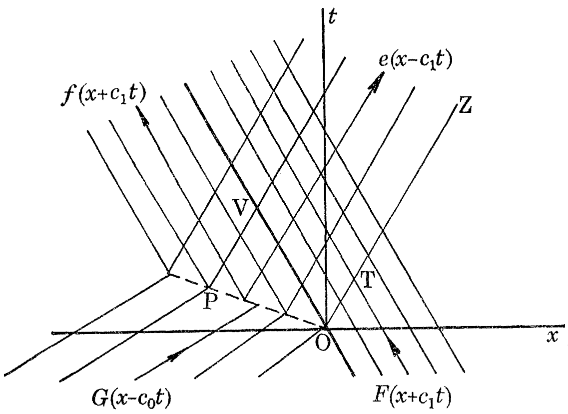

FIGURE 9. Characteristics diagram for meeting interaction with super-fast elastic-plastic interface, type F; interface path shown ---.

5. VALIDITY AND UNIQUENESS OF THE INITIAL INTERACTION SOLUTION

The formal solutions obtained in §4, assuming in turn interfaces of types C, B, D, E and F, will now be examined for validity. That is, the interface velocity must be in the assumed range, the elastic stress distribution must not exceed yield, and the plastic stress distribution must not be unloading. In view of the implicit dependence of the wave functions on the interface path, the explicit behaviour of the elastic and plastic stress distributions in terms of arbitrary initial functions may be determined, in general, only by series expansions of the solution about points on the interface path. The validity conditions so deduced apply only to some finite open strip in the (x, t) plane which contains the interface path. An exception is the type C solution, for which conditions ensuring global validity are found. If the elastic or plastic stress distribution violates the validity requirements outside the strip, then a new interaction is initiated which may be treated separately. After a time the newly created waves may reach the original interface and so replace the initially prescribed profiles. In fact it is clear that the local solution is governed by waves currently incident on the interface, but the term 'initial waves' is convenient.

Behaviour of the solutions in the neighbourhood of a generic point on the interface path, labelled P in figures 5 to 9, involve derivatives of the initial wave functions and yield stress distribution evaluated at P. It is assumed that the solution is (locally) valid until that instant and the next time interval is considered. This is governed by disturbances propagating along subsequent incident characteristics on the two sides, so the changes are described respectively by the left- and right-hand derivatives of the elastic and plastic wave functions $G(x-c_0t)$, $F(x+c_1t)$. Thus the results obtained apply if derivatives are interpreted as the appropriate left- or right-hand limit. The existence of these limits is seen to be unduly restrictive but allows compact expressions for validity conditions, and in any case is a weak restriction from a physical standpoint. By considering a sequence of points P on the interface path separated by short time intervals, the solution may be investigated in some finite strip containing the interface path. In particular, the behaviour of the solution in the first interval of time, which starts with a valid stress distribution defined by the initial waves, is determined by derivatives evaluated at the initial point, labelled O.

For each of the five formal solutions, conditions necessary for local validity are obtained in terms of the two initial wave functions and the yield stress distribution. The three sets of conditions are found to be non-overlapping, so that only one of the solutions is possible for given initial functions, but further, there are initial functions which do not satisfy any of the sets, when no single-interface solution is valid. These conclusions apply to some initial time interval. As the interaction proceeds through successive time intervals the solution may change in type, and waves initiated in earlier stages may be currently incident on the interface. This corresponds to the general initial conditions, involving incident wave functions $H(x+c_0t)$, $E(x-c_1t)$, treated in §6, when an interface of type A also is shown to occur.

The above conditions specifically involve the first non-vanishing derivatives of the initial wave functions and yield stress distribution evaluated at the point P on the interface path, and it is useful to introduce a short notation. More precisely, the value of the first derivative, for example F'_P, is ignored, and M is the least integer $\geqslant 2$ for which the Mth derivative $F^{(M)} \neq 0$. The suffix P will also be dropped when the expansion point is clear. A similar notation, $G^{(N)}$, $Y^{(S)}$, is applied to the other functions, where M, N, S may take different values, but in the case of equal values the same letter is used.

5·1. *Validity conditions for the solution of type C*

The elastic and plastic stress distributions are given by (4·1·2), (4·1·10), and (4·1·3), (4·1·11), respectively, and the interface path by (4·1·13). The interface velocity, given by (4·1·14), must satisfy $0 < \dot{X} < c_1$.

Consider first the plastic region $x > X(t)$ in which the stress must nowhere unload. Referring to figure 5, this condition is automatically satisfied in the domain COZ containing only the initial loading wave, leaving the domain bounded by OP and the first reflected characteristic OZ to examine. At a typical point V, differentiating (4·1·3) with respect to t,

$$\frac{1}{c_1}\left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{V}} = -e'_{\mathrm{V}} + F'_{\mathrm{V}}. \tag{5·1·1}$$

By the characteristic relations and (4·1·12), $F'_{\mathrm{V}} = F'_{\mathrm{D}}$, $e'_{\mathrm{V}} = e'_{\mathrm{P}} = F'_{\mathrm{P}} = F'_{\mathrm{C}}$, so that

$$\frac{1}{c_1}\left(\frac{\partial \sigma}{\partial t}\right)_{\mathrm{V}} = F'_{\mathrm{D}} - F'_{\mathrm{C}}. \tag{5·1·2}$$

Thus $(\partial\sigma/\partial t)_{\mathrm{V}} \geqslant 0$ as $\mathrm{V} \to \mathrm{P}$ if $F'_{\mathrm{D}} \geqslant F'_{\mathrm{C}}$ as $\mathrm{D} \to \mathrm{C}$, equivalent to

$$F''_{\mathrm{C}} = F''_{\mathrm{P}} \geqslant 0, \quad F^{(M)}_{\mathrm{P}} > 0. \tag{5·1·3}$$

If $F^{(m)}_{\mathrm{P}} = 0$, all $m \geqslant 2$, then the 'no unloading' criterion is still satisfied. For a valid solution in COP it is necessary and sufficient that

$$F^{(M)} > 0 \tag{5·1·4}$$

at every point on OP, or that all derivatives from the second vanish. This is a global result, noting that the value of M can change between points on OP. In particular, for a valid plastic solution in some neighbourhood of the interface during an initial time interval, defined by a region COP as $\mathrm{P} \to \mathrm{O}$, the condition is that $F''_{\mathrm{P}} \geqslant 0$ as $\mathrm{P} \to \mathrm{O}$, equivalent to (5·1·4) applied at O.

It is convenient to consider next the interface velocity $\dot{X}$. Rearranging (4·1·14) shows that

$$(c_0 - \dot{X})\, G'' = -(c_1 + \dot{X})\, F'' \tag{5·1·5}$$

at each point P on the interface path. By (5·1·3) and the inequality $0 < \dot{X} < c_1$, it follows that

$$G'' \leqslant 0. \tag{5·1·6}$$

Excluding for the moment the case when F'' and G'' vanish together, (5·1·5) gives the expressions

$$\dot{X} = \frac{c_0 G'' + c_1 F''}{G'' - F''}, \quad c_1 - \dot{X} = -\frac{(c_0 - c_1)\, G'' + 2c_1 F''}{G'' - F''}, \tag{5·1·7}$$

which, in association with (5·1·3) and (5·1·6), show that the necessary and sufficient condition for $0 < \dot{X} < c_1$ is

$$-(c_0 - c_1)\, G'' < 2c_1 F'' < -2c_0 G''. \tag{5·1·8}$$

If, however, F'', G'' vanish together at a point on the interface path, repeated differentiation of (4·1·13) until the first non-vanishing derivative of F or G at that points occurs, shows that

$$(c_0 - \dot{X})^{m-1}\,(-1)^{m-1}\, G^{(m)} = (c_1 + \dot{X})^{m-1}\, F^{(m)}. \tag{5·1·9}$$

The special linear profiles case with, by (4·1·13), $G' \equiv F'$ and all higher derivatives vanishing, is relevant also to the other types of solution, and is examined separately in appendix B. For a compatible $\dot{X}$ both $G^{(m)}$ and $F^{(m)}$ must be non-zero, so that $m = M = N$, and recalling (5·1·3),

$$F^{(M)} > 0, \quad (-1)^{M-1}\, G^{(M)} > 0. \tag{5·1·10}$$

Further, it is necessary and sufficient for $0 < \dot{X} < c_1$ that

$$\frac{c_1}{c_0} < \frac{c_1+\dot{X}}{c_0-\dot{X}} < \frac{2c_1}{c_0-c_1}, \tag{5·1·11}$$

which, by (5·1·9) and (5·1·10) is equivalent to

$$(c_0-c_1)^{M-1}(-1)^{M-1}G^{(M)} < (2c_1)^{M-1}F^{(M)} < (2c_0)^{M-1}(-1)^{M-1}G^{(M)}. \tag{5·1·12}$$

The previous condition (5·1·8) is just the case $M = 2$, but is given as an explicit example since it represents the most common situation in practice. In particular, (5·1·12), or the special modification discussed below, must hold at O.

Suppose now that the formal solution gives instantaneously an interface velocity at one end of the range, here $\dot{X} = 0$ or $\dot{X} = c_1$. Both these limiting cases are excluded from the range, but the solution will apply to some subsequent time interval provided that the first non-vanishing derivative of $\dot{X}$ at that instant has the appropriate sign. Validity depends further on the stress distributions. The left-hand limit of the range, $\dot{X} = 0$, is obtained if equality between the latter two expressions in (5·1·12) occurs, and the right-hand limit, $\dot{X} = c_1$, if equality between the first two expressions. The following notation allows more compact expression of the general condition (5·1·12), and of the end-point conditions. Define

$$\left.\begin{aligned}\mathscr{C}\langle m\rangle &= (2c_0)^{m-1}(-1)^{m-1}G^{(m)} - (2c_1)^{m-1}F^{(m)},\\ \mathscr{B}\langle m\rangle &= (c_0-c_1)^{m-1}(-1)^{m-1}G^{(m)} - (2c_1)^{m-1}F^{(m)},\end{aligned}\right\} \tag{5·1·13}$$

then (5·1·12) becomes

$$\mathscr{C}\langle M\rangle > 0, \quad \mathscr{B}\langle M\rangle < 0. \tag{5·1·14}$$

The interface velocity limits are given by

$$\dot{X} = 0 \Leftrightarrow \mathscr{C}\langle M\rangle = 0, \quad \dot{X} = c_1 \Leftrightarrow \mathscr{B}\langle M\rangle = 0. \tag{5·1·15}$$

Conditions governing the interface acceleration at end-points of the velocity range are analogous for each type of solution, and are determined in appendix C after reduction to common form. In this solution

$$\begin{aligned}\dot{X} = 0 \Leftrightarrow \mathscr{C}\langle M\rangle = 0; \quad (\dot{X})^{(r)} = 0 \Leftrightarrow \mathscr{C}\langle M+r\rangle = 0 \quad (r = 1, \ldots, m-1),\\ (\dot{X})^{(m)} > 0 \Leftrightarrow \mathscr{C}\langle M+m\rangle > 0,\end{aligned} \tag{5·1·16}$$

$$\begin{aligned}\dot{X} = c_1 \Leftrightarrow \mathscr{B}\langle M\rangle = 0; \quad (\dot{X})^{(r)} = 0 \Leftrightarrow \mathscr{B}\langle M+r\rangle = 0 \quad (r = 1, \ldots, m-1),\\ (\dot{X})^{(m)} < 0 \Leftrightarrow \mathscr{B}\langle M+m\rangle < 0.\end{aligned} \tag{5·1·17}$$

The elastic region $x < X(t)$ splits naturally into the two domains AON and NOP, in figure 5. The stress in AON is defined by the initial elastic wave $G(x-c_0t)$, where by (4·1·13) and (4·4), $G' = F' \geqslant 0$, and further, if $G'_0 = 0$, (5·1·10) applies at O. Thus the wave front is strictly unloading, and in view of (4·5), there is some finite neighbourhood of O, bounded by AO and ON, in which $G(x-c_0t) \leqslant Y(x)$ as required. If the elastic wave is totally unloading then this result is global, holding within the entire domain AON defined by the initial wave. The requirement $\sigma(x,t) \leqslant Y(x)$ in the domain NOP must now be considered. In fact the conditions for validity already obtained are shown to imply $\partial\sigma/\partial t \leqslant 0$ in some finite strip adjacent to the interface path. This is a stronger condition, since for particles in $x < 0$ the stress has been shown not to exceed yield on the limiting characteristic

ON (at least on some section adjacent to O), while in $x \geqslant 0$ the particles are just at yield on the interface path OP.

At a typical point K, by (4·1·2) and the characteristic relations,

$$\frac{1}{c_0}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{K}} = h'_{\mathrm{P}} - G'_{\mathrm{A}}. \tag{5·1·18}$$

Further, the vanishing of $(\partial\sigma/\partial t)_e$ on the interface, (3·9), shows that

$$h'_{\mathrm{P}} = G'_{\mathrm{P}}, \tag{5·1·19}$$

so finally

$$\frac{1}{c_0}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{K}} = G'_{\mathrm{B}} - G'_{\mathrm{A}}. \tag{5·1·20}$$

Thus if (5·1·6) holds on OP, say, then $(\partial\sigma/\partial t)_{\mathrm{K}} \leqslant 0$ for all points K up to the limiting characteristic BP (within NOP) which is a global result. More particularly, (5·1·10) applied at O implies (5·1·6) holds over an adjacent section and ensures validity in a finite neighbourhood of O.

The above results are summarized in § 5·6 as part of the uniqueness proof.

5·2. *Validity conditions for the solution of type B*

The elastic and plastic stress distributions are given by (4·1·2), (4·2·2) and (4·1·3) with $e(x) \equiv 0$, respectively. The interface path and velocity are given by (4·2·1) and (4·2·3), and $\dot{X}$ must satisfy $c_1 \leqslant \dot{X} \leqslant c_0$.

In the plastic region $x > X(t)$ the stress is defined solely by the initial plastic loading wave, since there is no reflected wave, and the 'no unloading' condition is automatically satisfied. The initial plastic loading condition (4·4) gives a further result

$$F' = 0 \Rightarrow F^{(M)} > 0. \tag{5·2·1}$$

A necessary condition on the elastic side of the interface is $(\partial\sigma/\partial t)_e \leqslant 0$, by (3·8), and so from (4·1·2), (4·2·2),

$$\frac{1}{c_0}\left(\frac{\partial\sigma}{\partial t}\right) = -\frac{2c_0^2(\dot{X}-c_1)\,G'}{(c_0-c_1)\,(c_0+\dot{X})} \leqslant 0. \tag{5·2·2}$$

This implies that if $\dot{X} \neq c_1$,

$$G' \geqslant 0. \tag{5·2·3}$$

It is now convenient to consider the interface velocity. Setting $\dot{X} = c_1$ in (4·2·3), and noting $F' \geqslant 0$, implies $G' \geqslant 0$, so that (5·2·3) holds for all $\dot{X}$ in the range $c_1 \leqslant \dot{X} \leqslant c_0$.

Rearranging (4·2·3) shows that

$$\left.\begin{aligned}(\dot{X}-c_1)\,\{2c_1G'+(c_0-c_1)\,F'\} &= 2c_1(c_0-c_1)\,(G'-F'),\\ (c_0-\dot{X})\,\{2c_1G'+(c_0-c_1)\,F'\} &= (c_0^2-c_1^2)\,F'.\end{aligned}\right\} \tag{5·2·4}$$

Thus, from the first relation,

$$G' \geqslant F', \quad \geqslant 0, \tag{5·2·5}$$

and further,

$$G' > F' \Rightarrow \dot{X} > c_1, \quad \dot{X} = c_1 \Rightarrow G' = F', \quad G' = F' \neq 0 \Rightarrow \dot{X} = c_1. \tag{5·2·6}$$

From the second relation $\dot{X} \leqslant c_0$ as required and

$$\dot{X} = c_0 \Rightarrow F' = 0, \quad G' > F' = 0 \Rightarrow \dot{X} = c_0. \tag{5·2·7}$$

If $G' = F' = 0$, $\dot{X}$ is not defined by (5·2·4), but repeated differentiation of (4·2·1) gives

$$2c_1(c_0-\dot{X})^m(-1)^{m-1}G^{(m)} = (c_0-c_1)(c_1+\dot{X})^m F^{(m)}, \tag{5·2·8}$$

where $G^{(m)}$ or $F^{(m)}$ (or both) is the first non-vanishing derivative. At least one exists for G', $F' \neq 0$. Immediately

$$\dot{X} = c_0 \Leftrightarrow N < M, \tag{5·2·9}$$

that is $G^{(N)} \neq 0$, $F^{(N)} = 0$; the sign of $G^{(N)}$ is determined later. For $\dot{X} < c_0$, $N = M$, and recalling (5·2·1),

$$F^{(M)} > 0, \quad (-1)^{M-1}G^{(M)} > 0. \tag{5·2·10}$$

Further, $\dot{X} \geqslant c_1$ implies

$$\frac{\dot{X}+c_1}{c_0-\dot{X}} \geqslant \frac{2c_1}{c_0-c_1}, \tag{5·2·11}$$

and it follows from (5·2·8), recalling the definition (5·1·13), that

$$\mathscr{B}\langle M\rangle \geqslant 0, \quad \dot{X} = c_1 \Leftrightarrow \mathscr{B}\langle M\rangle = 0. \tag{5·2·12}$$

Note that (5·2·5) and (5·2·12) allow G′ or $(-1)^{M-1}G^{(M)} \to \infty$ respectively, so that the existence of the (left-hand) derivative is not essential.

Before considering the further conditions required at the end-points $\dot{X} = c_1$, c_0, the sign of $G^{(N)}$ in the case $F' = G' = 0$, $\dot{X} = c_0$, described by (5·2·9), is needed. With $G' = 0$, (5·2·2) does not show that σ is decreasing and it is necessary that the first non-vanishing derivative on the elastic side is strictly negative. By (4·2·2), successive derivatives of h vanish with the corresponding derivatives of G at an interface point, so that the first non-vanishing derivative of (4·1·2), on eliminating $h^{(N)}$, is

$$-\frac{1}{c_0}\left(\frac{\partial^N\sigma}{\partial t^N}\right) = (-1)^{N-1}G^{(N)}\left\{1-\frac{c_0+c_1}{c_0-c_1}\left(\frac{c_0-\dot{X}}{c_0+\dot{X}}\right)^N\right\}. \tag{5·2·13}$$

Thus, since $\{\ \} > 0$,

$$(-1)^{N-1}G^{(N)} > 0. \tag{5·2·14}$$

Conditions so far determined to ensure elastic unloading at the interface path, that is for particles in $x \geqslant 0$, are $G' > F'$ or (5·2·14) if $F' = G' = 0$, leaving the case $F' = G' \neq 0$, when $\dot{X} = c_1$ and $(\partial\sigma/\partial t)_e = 0$. No simple expression analogous to (5·2·13) is obtained with $G' \neq 0$. However, if this situation first occurs at P, say, it is sufficient to show that $(\partial\sigma/\partial t)$ is strictly negative ahead of P on the interface path and at K (as K→P) on the reflected characteristic from P. Then by continuity, $(\partial\sigma/\partial t) < 0$ in a neighbourhood of P bounded by PK and the subsequent interface path, and the yield stress is not exceeded; it is assumed that conditions on OP meet the validity requirement up to PK. This situation may occur at O. The first condition is met if $\dot{X}$ increases from c_1 ahead of P, since then $\dot{X} > c_1$, $G' > 0$ in (5·2·2), and this is an end-point requirement determined shortly. Now, with $\dot{X} = c_1$ at P,

$$-\frac{1}{c_0}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{K}} = G'_{\mathrm{A}} - G'_{\mathrm{B}}, \tag{5·2·15}$$

which is positive as A→B, as required, in view of (5·2·14).

For particles in $x < 0$, the stress in the domain AON is defined by the initial wave, again strictly unloading at the front by (5·2·5), (5·2·14), so yield is not exceeded near O. Then, for the domain beyond ON, it is sufficient to show that $(\partial\sigma/\partial t)_{\mathrm{L}} < 0$ as L→O along NO. This follows immediately by continuity if $(\partial\sigma/\partial t)_{\mathrm{L}} < 0$, and if $(\partial\sigma/\partial t)_{\mathrm{o}} = 0$ the above argument with KP replaced by LO leads to (5·2·14) again, applied at O.

Finally, some examination of the limit situations $\dot{X} = c_1$ or $\dot{X} = c_0$, here permissible, is required. We exclude general discussion of the particular cases $\dot{X} \equiv c_1$ and $\dot{X} \equiv c_0$, since the predetermined interface path allows explicit solutions, necessarily of type B. Particular solutions of this form are obtained in appendix B in the special linear profiles case, and it is assumed here that at least one $F^{(M)}, G^{(N)} \neq 0$ $(M, N \geqslant 2)$. Thus if $\dot{X} = c_1$ instantaneously, a continuing solution of type B requires $\dot{X}$ to increase, while if $\dot{X} = c_0$ instantaneously, a continuing solution requires $\dot{X}$ to decrease. The following conditions are derived in appendix C, noting that all situations are covered by including non-zero first derivatives as $F^{(M)}, G^{(N)}$ $(M, N = 1)$.

$$\left.\begin{aligned} \dot{X} = c_1 \Leftrightarrow \mathscr{B}\langle M\rangle = 0; \quad (\dot{X})^{(r)} = 0 \Leftrightarrow \mathscr{B}\langle M+r\rangle = 0 \quad (r = 1, \ldots, m-1),\\ \dot{X}^{(m)} > 0 \Leftrightarrow \mathscr{B}\langle M+m\rangle > 0; \end{aligned}\right\} \tag{5·2·16}$$

$$\left.\begin{aligned} \dot{X} = c_0 \Leftrightarrow F^{(N)} = 0; \quad (\dot{X})^{(r)} = 0 \Leftrightarrow F^{(N+r)} = 0 \quad (r = 1, \ldots, M-N-1),\\ \dot{X}^{(M-N)} < 0 \Leftrightarrow F^{(M)} > 0. \end{aligned}\right\} \tag{5·2·17}$$

It follows from (5·2·1) that (5·2·17) is necessarily satisfied. The limit situation in (5·2·16) arose in the type C solution, but the comparison and complete summary of conditions is deferred until §5·6.

5·3. *Validity conditions for the solution of type D*

The elastic and plastic stress distributions are given by (4·1·2), (4·3·1) and (4·1·3), (4·1·11) respectively. The interface path and velocity are given by (4·3·2) and (4·3·3), and $\dot{X}$ must satisfy $-c_1 \leqslant \dot{X} \leqslant 0$.

In the plastic region $x > X(t)$ there is a valid stress distribution in the domain DOZ (figure 7) defined by the initial plastic loading wave. The loading condition (3·10) must be satisfied at each point P on the interface path. From (4·1·3) and (4·1·11),

$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right) = \frac{2c_1(c_0-\dot{X})}{(c_0+c_1)\,(c_1-\dot{X})}(F'-G') \geqslant 0, \tag{5·3·1}$$

so it is necessary that

$$F' \geqslant G'. \tag{5·3·2}$$

If $F' > G'$ then $(\partial\sigma/\partial t) > 0$, but if $F' = G'$, at P say, then to ensure loading in a neighbourhood of the interface as the interaction proceeds it is necessary and sufficient that $F'-G'$ becomes positive beyond P, and that $(\partial\sigma/\partial t)_{\mathrm{V}} > 0$ as $\mathrm{V} \to \mathrm{P}$ along the reflected characteristic from P. From (4·1·11), $G'_{\mathrm{P}} = F'_{\mathrm{P}} \Rightarrow e'_{\mathrm{P}} = F'_{\mathrm{P}}$, whence

$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{V}} = F'_{\mathrm{D}} - F'_{\mathrm{C}}, \tag{5·3·3}$$

with the consequent requirement

$$F' = G' \Rightarrow F^{(M)} > 0. \tag{5·3·4}$$

In particular, for validity in the initial time interval it is necessary that either $F' > G'$, or that (5·3·4) holds at O.

If $F' = G'$ and $\dot{X} = 0$ at P, then it is shown in appendix C that $F'-G'$ increasing and $\dot{X}$ decreasing are equivalent, and the necessary and sufficient conditions are given in the interface velocity discussion. However, it is seen there that $\dot{X} = 0$ does not necessarily follow from $F' = G'$, and then it is possible only to obtain necessary conditions when $F' - G'$ increases. Consider a point M on the interface path at a small time interval δt beyond P, then the leading term expansions about P give

$$\left.\begin{aligned} F'_{\mathrm{M}} - F'_{\mathrm{P}} &= (c_1 + \dot{X})^{M-1} F_{\mathrm{P}}^{(M)} (\delta t)^{M-1}/(M-1)!, \\ G'_{\mathrm{M}} - G'_{\mathrm{P}} &= (c_0 - \dot{X})^{N-1} (-1)^{N-1} G^{(N)} (\delta t)^{N-1}/(N-1)!. \end{aligned}\right\} \tag{5·3·5}$$

Thus since $F'_{\mathrm{P}} = G'_{\mathrm{P}}$ and $F_{\mathrm{P}}^{(M)} > 0$ it follows that $F'_{\mathrm{M}} > G'_{\mathrm{M}}$ if, and only if, $M < N(\dot{X} \neq -c_1)$ or

$$M > N \quad \text{or} \quad M \leqslant N \quad \text{and} \quad \dot{X} = -c_1 \Rightarrow (-1)^N G^{(N)} > 0, \tag{5·3·6}$$

$$M = N, \quad \dot{X} \neq -c_1 \Rightarrow (c_1 + \dot{X})^{M-1} F^{(M)} > (c_0 - \dot{X})^{M-1} (-1)^{M-1} G^{(M)}. \tag{5·3·7}$$

The latter is trivially satisfied if $(-1)^M G^{(M)} > 0$. If $(-1)^{M-1} G^{(M)} > 0$, it is necessary that the inequality holds when the l.h.s. takes its maximum value and the r.h.s. its minimum value for $-c_1 \leqslant \dot{X} \leqslant 0$, that is when $\dot{X} = 0$. In terms of the definition (5·1·13) this condition becomes

$$(-1)^{M-1} G^{(M)} > 0 \Rightarrow \mathscr{C}\langle M \rangle < 0, \tag{5·3·8}$$

and is not sufficient when $\dot{X} < 0$. If the equality in (5·3·7) holds, then the first corresponding strict inequality in the higher derivatives must have the same sign. The special case $F' = G'$ and $F^{(m)}, G^{(m)} = 0$ $(m \geqslant 2)$ is treated in appendix B.

In the elastic region $x < X(t)$ all particles are just at yield on the interface path, and the loading condition (3·10) must be satisfied at each point P. From (4·1·2), eliminating h and Y in turn by (4·3·1) and (4·3·2),

$$\frac{1}{c_0}\left(\frac{\partial \sigma}{\partial t}\right) = \frac{-\dot{X}}{c_0 + \dot{X}} (2G' - Y') = \frac{2c_0(c_1 + \dot{X})}{(c_0 + c_1)(c_0 + \dot{X})} (F' - G'), \tag{5·3·9}$$

which is non-negative by (5·3·2). It further implies that

$$2G' \geqslant Y', \tag{5·3·10}$$

except perhaps if $\dot{X} = 0$, but it will be shown that (5·3·10) is always necessary. A strict loading condition from (5·3·9) depends on the value of $\dot{X}$, but is clearly ensured if the strict inequalities of both (5·3·2) and (5·3·10) apply. Again, if $F' = G'$ and $\dot{X} \neq 0$, then by (5·3·9) $2G' = Y'$, and a valid solution certainly requires that both $F' - G'$ and $2G' - Y'$ increase from zero in the subsequent time interval. The former requirement has already been treated. In addition it is necessary that $Y_{\mathrm{K}} - \sigma_{\mathrm{K}} \geqslant 0$ as $\mathrm{K} \to \mathrm{P}$ along the reflected characteristic.

Let $x_{\mathrm{K}} = x_{\mathrm{P}} - \delta x$, then eliminating $h_{\mathrm{K}} = h_{\mathrm{P}}$ by (4·3·1) and retaining the two leading terms in expansions about P shows that

$$Y_{\mathrm{K}} - \sigma_{\mathrm{K}} = (2G'_{\mathrm{P}} - Y'_{\mathrm{P}})\,\delta x + 2^N (-1)^{N-1} G_{\mathrm{P}}^{(N)} (\delta x)^N/N! + (-1)^S Y_{\mathrm{P}}^{(S)} (\delta x)^S/S!. \tag{5·3·11}$$

Immediately the inequality (5·3·10) is confirmed for $\dot{X} = 0$. If $2G'_{\mathrm{P}} = Y'_{\mathrm{P}}$ the alternative requirements are

$$\left.\begin{aligned} &S < N \Rightarrow (-1)^S Y^{(S)} > 0, \quad S > N \Rightarrow (-1)^{N-1} G^{(N)} > 0, \\ &S = N \Rightarrow (-1)^N Y^{(N)} > 0, \quad (-1)^{N-1} G^{(N)} > 0, \end{aligned}\right\} \tag{5·3·12}$$

$$\left.\begin{aligned} \text{or} \quad & (-1)^N Y^{(N)} \geqslant 2^N (-1)^N G^{(N)} > 0, \\ \text{or} \quad & 2^N (-1)^{N-1} G^{(N)} \geqslant (-1)^{N-1} Y^{(N)} > 0. \end{aligned}\right\} \tag{5·3·13}$$

In the domain AON containing only the initial elastic wave, the stress does not exceed yield near O if $G'_O > 0$, or if $G'_O = 0$ and $(-1)^{N-1} G_O^{(N)} > 0$, since $Y'_O \leqslant 0$. If $G'_O = 0$ and $(-1)^{N-1} G_O^{(N)} < 0$, the stress remains below yield if $Y'_O < 0$, or if $Y'_O = 0$ and $S < N$, since $(-1)^S Y^{(S)} > 0$, but exceeds yield if $S > N$. Thus (5·3·12) is again a requirement at O if $G'_O = Y'_O = 0$. Also in this case, with $S = N$, the first set of conditions (5·3·13) trivially ensures validity, while the third set is not possible at O, and a similar expansion procedure shows that the second set is again the requirement when $(-1)^N G_O^{(N)} > 0$. That is, (5·3·12), (5·3·13) cover all necessary alternatives when $G'_O = Y'_O = 0$.

Given $2G'_P = Y'_P$, then at the point M, a time δt later, the leading term expansions give

$$2G'_M - Y'_M = 2(c_0 - \dot{X})^{N-1} (-1)^{N-1} G_P^{(N)} (\delta t)^{N-1}/(N-1)! + (-\dot{X})^{S-1} (-1)^S Y^{(S)} (\delta t)^{S-1}/(S-1)!, \tag{5·3·14}$$

which is required to be positive. Thus, for any S, N,

$$\dot{X} = 0 \Rightarrow (-1)^{N-1} G^{(N)} > 0, \tag{5·3·15}$$

when (5·3·6) and the second set of conditions in (5·3·12) are not permissible. For $\dot{X} \neq 0$, the conditions (5·3·12) and the first set of (5·3·13) again follow. If $(-1)^{N-1} G^{(N)}$, $(-1)^{N-1} Y^{(N)} > 0$, the third set of (5·3·13) is already a stronger requirement, since

$$\frac{c_0 - \dot{X}}{-\dot{X}} > \frac{c_0 + c_1}{c_1} > 2. \tag{5·3·16}$$

But in place of the second set a stronger condition is necessary,

$$(-1)^N Y^{(N)}, \quad (-1)^N G^{(N)} > 0; \quad \mathscr{D}\langle N \rangle \geqslant 0, \tag{5·3·17}$$

where
$$\mathscr{D}\langle m \rangle = (c_1)^{m-1} (-1)^m Y^{(m)} - 2(c_0 + c_1)^{m-1} (-1)^m G^{(m)}. \tag{5·3·18}$$

This is obtained when $(-\dot{X})/(c_0 - \dot{X})$ takes its maximum value at $\dot{X} = -c_1$, and is not sufficient if $\dot{X} > -c_1$. If $\mathscr{D}\langle N \rangle = 0$, then the first corresponding strict inequality in the higher derivatives must have the same sign.

If at least one of the strict inequalities (5·3·2), (5·3·10) holds, then (4·3·3) defines an $\dot{X} \leqslant 0$ as required, but in all cases

$$\dot{X} = 0 \Rightarrow F' = G'. \tag{5·3·19}$$

Also, the rearrangement

$$\{2c_0 F' + 2c_1 G' - (c_0 + c_1) Y'\} (\dot{X} + c_1) = c_1(c_1 + c_0) (2G' - Y') \tag{5·3·20}$$

shows that $\dot{X} \geqslant -c_1$ if one strict inequality holds, and in general

$$\dot{X} = -c_1 \Rightarrow 2G' = Y'. \tag{5·3·21}$$

The inverse implications of (5·3·19), (5·3·21) also follow except when $2F' = 2G' = Y'$. In the case $2F' = 2G' = Y'$ further differentiation of (4·3·2) is required to determine $\dot{X}$. Since $F'_O \geqslant 0$, $Y'_O \leqslant 0$, this occurs initially only if $F'_O = G'_O = Y'_O = 0$. However, at a later stage with new 'initial' conditions, involving an initial wave $H(x+c_0t)$, $Y' > 0$ is possible and the case $2F' = 2G' = Y' > 0$ may arise. This case must be included in a complete examination of a type D solution, and is required for later comparison with the other formal solutions to a general interaction obtained in §6.

First, consider the case $F' = G' = Y' = 0$, at P say. Repeated differentiation of (4·3·2) gives

$$2c_0(c_1+\dot{X})^m F^{(m)} + 2c_1(c_0-\dot{X})^m (-1)^m G^{(m)} - (c_0+c_1)(-\dot{X})^m (-1)^m Y^{(m)} = 0, \quad (5·3·22)$$

where the mth derivative is the lowest non-vanishing derivative of either F, G, or Y at the point P. If it is $Y^{(m)}$, then immediately

$$S < M,\ N \Rightarrow \dot{X} = 0, \quad (5·3·23)$$

which is a permissible end-point velocity of the range, together with the consequent requirement (5·3·15). Similarly

$$M < N,\ S \Rightarrow \dot{X} = -c_1, \quad (5·3·24)$$

the other end-point velocity, when (5·3·6) applies, but $N < M$, $S \Rightarrow \dot{X} = c_0$ and is not permitted. For $S = M < N$, (5·3·22) becomes

$$\frac{2c_0}{c_0+c_1}\left(\frac{c_1+\dot{X}}{-\dot{X}}\right)^M = \frac{(-1)^M Y^{(M)}}{F^{(M)}} \neq 0, \quad (5·3·25)$$

and in view of the inequality $-c_1 \leqslant \dot{X} \leqslant 0$, and $F^{(M)} > 0$,

$$S = M < N \Rightarrow \dot{X} \neq 0,\ -c_1; \quad (-1)^M Y^{(M)} > 0. \quad (5·3·26)$$

For $S = N < M$, recalling (5·3·6),

$$(c_0+c_1)(-\dot{X})^N (-1)^N Y^{(N)} = 2c_1(c_0-\dot{X})^N (-1)^N G^{(N)} > 0. \quad (5·3·27)$$

Thus, in view of the inequality $-c_1 \leqslant \dot{X} \leqslant 0$, it follows that

$$S = N < M \Rightarrow \dot{X} \neq 0, \quad \mathscr{D}\langle N\rangle \geqslant 0, \quad \text{or} \quad \dot{X} = -c_1 \Leftrightarrow \mathscr{D}\langle N\rangle = 0, \quad (5·3·28)$$

where the definition (5·3·18) is used. Similarly, for $S > M = N$, recalling $(-1)^{M-1} G^{(M)} > 0$ by (5·3·12), and the definition (5·1·13),

$$S > M = N \Rightarrow \dot{X} \neq -c_1; \quad \mathscr{C}\langle M\rangle \leqslant 0, \quad \dot{X} = 0 \Leftrightarrow \mathscr{C}\langle M\rangle = 0. \quad (5·3·29)$$

It remains to consider the case $S = M = N$ for which each term in (5·3·22) is non zero, except when $\dot{X} = 0$ or $-c_1$. The conclusions depend on the signs of $G^{(M)}$, $Y^{(M)}$. If $(-1)^{M-1} G^{(M)} > 0$ and $(-1)^M Y^{(M)} > 0$, then

$$0 < \frac{(-1)^{M-1} G^{(M)}}{F^{(M)}} \leqslant \frac{c_0(c_1+\dot{X})^M}{c_1(c_0-\dot{X})^M}, \quad (5·3·30)$$

with the equality holding if, and only if, $\dot{X} = 0$, so that (5·3·29) again applies. If $(-1)^{M-1} G^{(M)} > 0$ and $(-1)^M Y^{(M)} < 0$, then the latter inequality of (5·3·30) is reversed, and no further restriction follows from the compatibility condition $-c_1 \leqslant \dot{X} \leqslant 0$. But here the necessary condition (5·3·8) shows that $\mathscr{C}\langle M\rangle \leqslant 0$, again. If $(-1)^M G^{(M)} > 0$, then necessarily from (5·3·22), $\dot{X} \neq 0$ and $(-1)^M Y^{(M)} > 0$, so that

$$\frac{(-1)^M Y^{(M)}}{(-1)^M G^{(M)}} \geqslant \frac{2c_1}{c_0+c_1}\left(\frac{c_0-\dot{X}}{-\dot{X}}\right)^M, \tag{5·3·31}$$

with the equality holding if, and only if, $\dot{X} = -c_1$. Thus (5·3·17) again follows, with $N = M$, together with an end-point velocity condition, namely

$$\mathscr{D}\langle M\rangle \geqslant 0, \quad \dot{X} = -c_1 \Leftrightarrow \mathscr{D}\langle M\rangle = 0. \tag{5·3·32}$$

Returning to the case $2F' = 2G' = Y' > 0$, no simple equation for $\dot{X}$ in terms of higher derivatives follows by further differentiation of (4·3·2). Thus no explicit compatibility conditions for $\dot{X}$ are obtained. However, the various conditions necessary for valid elastic and plastic stress distributions, obtained prior to (5·3·22) are shown to rule out valid solutions of other types. Details are presented in the uniqueness and existence discussion in the next section.

Finally, it has been noted in the stress validity discussions that if $\mathscr{D}\langle M\rangle = 0$ or $\mathscr{C}\langle M\rangle = 0$, then the first corresponding strict inequalities in the higher derivatives must have the appropriate sign, that is $\mathscr{D}\langle M+m\rangle > 0$, $\mathscr{C}\langle M+m\rangle < 0$. When also the vanishing of $\mathscr{D}\langle M\rangle$ or $\mathscr{C}\langle M\rangle$ is accompanied by $\dot{X} = -c_1$ or $\dot{X} = 0$ respectively, as in (5·3·28), (5·3·32), and (5·3·29), then it is shown in appendix C that the above strict inequalities respectively ensure that $\dot{X}$ increases or decreases. That is

$$\left.\begin{aligned} \dot{X} = -c_1, \quad \mathscr{D}\langle M\rangle = 0; \quad (\dot{X})^{(r)} = 0 \Leftrightarrow \mathscr{D}\langle M+r\rangle = 0 \quad (r = 1, \ldots, m-1),\\ \dot{X}^{(m)} > 0 \Leftrightarrow \mathscr{D}\langle M+m\rangle > 0; \end{aligned}\right\} \tag{5·3·33}$$

$$\left.\begin{aligned} \dot{X} = 0, \quad \mathscr{C}\langle M\rangle = 0; \quad (\dot{X})^{(r)} = 0 \Leftrightarrow \mathscr{C}\langle M+r\rangle = 0 \quad (r = 1, \ldots, m-1),\\ (\dot{X})^{(m)} < 0 \Leftrightarrow \mathscr{C}\langle M+m\rangle < 0. \end{aligned}\right\} \tag{5·3·34}$$

These results apply also in the case $2F' = 2G' = Y' > 0$, although here the first equality does not itself necessarily imply the appropriate end-point velocity. Furthermore, they apply to the cases $2F' > 2G' = Y'$, when $\dot{X} = -c_1$ by (5·3·21), and $2F' = 2G' > Y'$, when $\dot{X} = 0$ by (5·3·19), by setting $M = 1$.

A summary of results is given in §5·6.

5·4. *Validity conditions for the solution of type E*

The elastic and plastic stress distributions are given by (4·1·2), (4·1·10) and (4·1·3), (4·4·2), (4·4·3) respectively. The interface path and velocity are given by (4·4·4) and (4·4·5), and $\dot{X}$ must satisfy $-c_1 > \dot{X} > -c_0$.

Since $\partial\sigma/\partial t$ vanishes on OP (figure 8) by the interface conditions (3·11), validity of the plastic stress distribution requires that the stress does not unload in some neighbourhood of O within the domain POZ, bounded by the first reflected characteristic OZ. This requires that $(\partial\sigma/\partial t)_{\mathrm{T}} > 0$ as T→O along OZ, and that $(\partial\sigma/\partial t)_{\mathrm{V}} > 0$ as P→O, where V is taken in turn to lie on the two sides of the dividing characteristic between the F and f domains. Using (4·4·3), it follows that

$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{T}} = F'_{\mathrm{T}} - G'_{\mathrm{O}}, \tag{5·4·1}$$

and letting T→O the consequent requirement at O is

$$F' \geqslant G', \quad F' = G' \Rightarrow F^{(M)} > 0. \tag{5·4·2}$$

Also
$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{V}} = \begin{cases} F_0' - G_{\mathrm{P}}' & (x = (-c_1 t)+), \\ f_0' - f_{\mathrm{P}}' & (x = (-c_1 t)-). \end{cases} \tag{5·4·3}$$

Letting P→O the requirement on $x = (-c_1 t)+$ is met in view of (5·4·2), while on $x = (-c_1 t)-$ the further condition at O is that

$$(-1)^{m-1} f^{(m)} < 0, \tag{5·4·4}$$

$f^{(m)}$ being the next non-vanishing derivative. But continued differentiation of (4·4·3) shows that

$$(\dot{X} + c_1)^{N-1} f^{(N)} = (\dot{X} - c_0)^{N-1} G^{(N)}, \tag{5·4·5}$$

and since $\dot{X} < -c_1$, in (5·4·4) $m = N$ and it follows that

$$(-1)^{N-1} G^{(N)} < 0. \tag{5·4·6}$$

The elastic stress distribution is just at yield on the interface path OP by (3·11), and $\partial\sigma/\partial t$ vanishes, so validity in a neighbourhood of O beyond the first reflected characteristic ON requires that $(\partial\sigma/\partial t)_{\mathrm{K}} > 0$ as K→O. Now using (4·4·3), it follows that

$$\frac{1}{c_0}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{K}} = -G_{\mathrm{P}}' + G_0', \tag{5·4·7}$$

which is positive as P→O(K→O) in view of (5·4·6). For validity near O up to ON it is necessary and sufficient that $Y(x) - G(x) > 0$, as $x \to 0$ through negative values. By (4·4·4) this is satisfied if $Y_0' < 0$ when $-Y_0' + G_0' > 0$, but for $Y_0' = G_0' = 0$ requires, anticipating the result $S = N$, that

$$(-1)^N Y^{(N)} > (-1)^N G^{(N)}. \tag{5·4·8}$$

Continued differentiation of (4·4·4) shows that

$$2(c_0 - \dot{X})^{N-1}(-1)^{N-1} G^{(N)} = (-\dot{X})^{N-1}(-1)^{N-1} Y^{(N)}, \tag{5·4·9}$$

where neither side vanishes since $-c_1 > \dot{X} > -c_0$, so that $S = N$ and, using (5·4·6),

$$(-1)^{N-1} Y^{(N)} < 0. \tag{5·4·10}$$

Since
$$2 < \frac{c_0 - \dot{X}}{-\dot{X}} < \frac{c_0 + c_1}{c_1}, \tag{5·4·11}$$

with the left- and right-hand equalities if $\dot{X} = -c_0$ and $-c_1$ respectively, it follows from (5·4·9) and (5·4·6), (5·4·10) that

$$\mathscr{D}\langle N\rangle < 0, \quad \mathscr{E}\langle N\rangle > 0, \tag{5·4·12}$$

recalling the definition (5·3·18) and defining

$$\mathscr{E}\langle m\rangle = (-1)^m Y^{(m)} - 2^m(-1)^m G^{(m)}. \tag{5·4·13}$$

Note that (5·4·13) is a stronger condition than (5·4·8). Further, by appendix C,

$$\left.\begin{array}{c}\dot{X} = -c_1 \Leftrightarrow \mathscr{D}\langle \mathrm{N}\rangle = 0; \quad (\dot{X})^{(r)} = 0 \Leftrightarrow \mathscr{D}\langle N+r\rangle = 0 \quad (r = 1, \ldots, m-1), \\ (\dot{X})^{(m)} < 0 \Leftrightarrow \mathscr{D}\langle N+m\rangle < 0;\end{array}\right\} \quad (5{\cdot}4{\cdot}14)$$

$$\left.\begin{array}{c}\dot{X} = -c_0 \Leftrightarrow \mathscr{E}\langle N\rangle = 0; \quad (\dot{X})^{(r)} = 0 \Leftrightarrow \mathscr{E}\langle N+r\rangle = 0 \quad (r = 1, \ldots, m-1), \\ (\dot{X})^{(m)} > 0 \Leftrightarrow \mathscr{E}\langle N+m\rangle > 0.\end{array}\right\} \quad (5{\cdot}4{\cdot}15)$$

Thus the first strict inequalities corresponding to (5·4·12), required for validity, are equivalent to the interface acceleration having the appropriate direction.

The summary of results is given in §5·6.

5·5. *Validity conditions for the solution of type F*

In the elastic region the stress is given simply by the initial elastic wave $G(x-c_0 t)$, and in the plastic region is given by (4·1·3) and (4·1·10), (4·1·11) with $h \equiv 0$ and F replaced by f. The interface path and velocity are given by (4·5·1) and (4·5·2), and $\dot{X}$ must satisfy $\dot{X} \leqslant -c_0$.

The validity conditions (3·12) require that the stress is loading on both elastic and plastic sides of the interface, thus

$$\left.\begin{array}{c}\dfrac{1}{c_0}\left(\dfrac{\partial\sigma}{\partial t}\right)_{\mathrm{e}} = -G' \geqslant 0, \\ \dfrac{1}{c_1}\left(\dfrac{\partial\sigma}{\partial t}\right)_{\mathrm{p}} = f'-e' = -\dfrac{c_1(c_0-\dot{X})\,(c_0+\dot{X})}{c_0(c_1-\dot{X})\,(c_1+\dot{X})}\,G' \geqslant 0,\end{array}\right\} \quad (5{\cdot}5{\cdot}1)$$

where (4·1·10), (4·1·11) are used to eliminate f', e'. It is therefore necessary and sufficient for elastic validity that $G' < 0$ or that $G' = 0$, at O say, and decreases along the interface path; hence

$$G' \leqslant 0, \quad G' = 0 \Rightarrow (-1)^N G^{(N)} > 0. \quad (5{\cdot}5{\cdot}2)$$

From (4·5·2), since $\dot{X} \leqslant -c_0$,

$$G' < 0 \Rightarrow G' - Y' \geqslant 0, \quad G' = 0 \Rightarrow Y' = 0, \quad (5{\cdot}5{\cdot}3)$$

while the rearrangement $\quad (\dot{X}+c_0)\,\{G'-Y'\} = -c_0(Y'-2G') \quad (5{\cdot}5{\cdot}4)$

then shows that $\quad 0 \geqslant Y' \geqslant 2G', \quad \dot{X} = -c_0 \Rightarrow Y' = 2G'. \quad (5{\cdot}5{\cdot}5)$

Furthermore, since $Y' = 2G' < 0 \Rightarrow G' - Y' > 0$, (4·5·2) shows that

$$Y' = 2G' < 0 \Rightarrow \dot{X} = -c_0. \quad (5{\cdot}5{\cdot}6)$$

If $G' = 0$ or $\dot{X} = -c_0$ then plastic validity at the interface, (5·5·1), requires respectively that G' decrease, given by (5·5·2), and that $\dot{X}$ decrease, equivalent to $Y'-2G'$ increasing by (5·5·5). Further, if either case occurs at O, then it is necessary that $(\partial\sigma/\partial t)_{\mathrm{V}} > 0$ as $\mathrm{V} \to \mathrm{O}$ along the first f characteristic. Now by (4·1·11), $G' = 0 \Rightarrow e' = f' = 0$, and $(-1)^N G^{(N)} > 0 \Rightarrow (-1)^N e^{(N)} > 0$, and hence

$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{V}} = f'_{\mathrm{O}} - e'_{\mathrm{P}} = -e'_{\mathrm{P}} > 0 \quad (5{\cdot}5{\cdot}7)$$

as required. At each interface point it follows from (4·1·10), (4·1·11), using $G' \leqslant 0$, that

$$f',\, e' \geqslant G', \quad f' = e' = G' \Leftrightarrow \dot{X} = -c_0. \quad (5{\cdot}5{\cdot}8)$$

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Thus in the case $\dot{X} = -c_0$ at O,

$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{V}} = G'_0 - e'_{\mathrm{P}} < G'_0 - G'_{\mathrm{P}}. \tag{5·5·9}$$

A necessary, but not sufficient, condition is then $G'_0 > G'_{\mathrm{P}}$, and therefore

$$\dot{X} = -c_0 \Rightarrow (-1)^N G^{(N)} > 0. \tag{5·5·10}$$

Thus, by (5·5·2), (5·5·6) and (5·5·10), always

$$Y' = 2G' \Rightarrow (-1)^N G^{(N)} > 0. \tag{5·5·11}$$

Plastic validity in the domain ZOP, in a neighbourhood of O, requires that $F'_0 > e'_0$ or that $F'_0 = e'_0$ and $F' - e'$ increases away from O within ZOP. Since $F' \geqslant 0$, $e' \leqslant 0$, the strict inequality holds unless $F'_0 = e'_0 = 0$, but then $F^{(M)} \geqslant 0$, $(-1)^N e^{(N)} > 0$ and $F' - e'$ increases as required. Necessarily $F'_0 \geqslant G'_0$ and $F'_0 = G'_0$ is possible only if both vanish, when $F^{(M)} > 0$, $(-1)^N G^{(N)} > 0$. In the case $F' > G'$, $2G' < Y' \leqslant 0$, both strict inequalities in (5·5·1) hold.

It remains to ensure that if $Y' - 2G' = 0$ at O then it increases in the subsequent time interval along the interface path. That is,

$$(-\dot{X})^{m-1}(-1)^{m-1} Y^{(m)} + 2(c_0 - \dot{X})^{m-1}(-1)^m G^{(m)} > 0, \tag{5·5·12}$$

where m is the least integer (> 1) giving a non-vanishing contribution. Immediately, (by 5·5·11), this is satisfied if $S > N$, while

$$S < N \Rightarrow (-1)^{S-1} Y^{(S)} > 0. \tag{5·5·13}$$

If $S = N$, (5·5·13) is a sufficient condition, leaving the alternative situation

$$\frac{(-1)^N G^{(N)}}{(-1)^N Y^{(N)}} > \frac{1}{2}\left(\frac{-\dot{X}}{c_0 - \dot{X}}\right)^{N-1} > 0. \tag{5·5·14}$$

If $Y' = 2G' < 0$, then by (5·5·6), recalling the definition (5·4·13), this becomes

$$S = N, \quad (-1)^N Y^{(N)} > 0, \quad \mathscr{E}\langle N\rangle < 0. \tag{5·5·15}$$

If $Y' = 2G' = 0$, continued differentiation of (4·5·1) shows that $S = N$ and

$$\left(\frac{c_0 - \dot{X}}{-\dot{X}}\right)^N = \frac{(-1)^N Y^{(N)}}{(-1)^N G^{(N)}}, \tag{5·5·16}$$

which shows that $\dot{X} = -c_0 \Leftrightarrow \mathscr{E}(N) = 0$, $\dot{X} < -c_0 \Leftrightarrow \mathscr{E}\langle N\rangle < 0$. On using (5·5·16), (5·5·14) reduces to $\dot{X} < -c_0$ which is therefore equivalent again to (5·5·15).

If $\mathscr{E}\langle N\rangle = 0$, then stress validity requires the first strict inequality corresponding to (5·5·15), which is shown in appendix C to be equivalent to $\dot{X}$ decreasing from $-c_0$. Here the case $N = 1$ is included in view of (5·5·6). Thus

$$\left.\begin{aligned} &\dot{X} = -c_0 \Leftrightarrow \mathscr{E}\langle N\rangle = 0; \quad (\dot{X})^r = 0 \Leftrightarrow \mathscr{E}\langle N+r\rangle = 0 \quad (r = 1, \ldots, m-1), \\ &(\dot{X})^{(m)} < 0 \Leftrightarrow \mathscr{E}\langle N+m\rangle < 0. \end{aligned}\right\} \tag{5·5·17}$$

The summary of results for each type of initial solution follows in the next section.

5·6. *Uniqueness*

The formal solutions of types C, B, D, E, F to the initial meeting interaction have now been examined in regard to validity within some finite region on both elastic and plastic sides of the interface. It is shown in §6 that the remaining type, A, is not possible initially. Under a wide range of conditions on the initial yield stress distribution and two initial wave functions, the existence of a valid solution of at least one type has been demonstrated over some subsequent interval of time. For various other conditions it is shown that none of these solutions is valid, that is no single-interface solution exists. Further, there are conditions which are shown to be necessary for validity of a given type of solution, but not sufficient to ensure validity. As discussed earlier, if no single-interface solution exists the only alternative is a multi-interface solution. In each solution the interface path is defined by an implicit algebraic equation. Again, for a wide range of initial conditions the Implicit Function Theorem guarantees the existence of a unique continuous solution, but for the other applicable conditions a modified theorem is needed. This is presented in appendix D, and all six types of interface path equation are discussed there.

It remains to show that the conditions necessary for the validity of each type of solution are mutually exclusive, so that for a given set of initial conditions only one type of solution can be valid in the subsequent time interval. Thus, when a single-interface solution does exist, it is unique, but this does not show that under the same set of initial conditions a valid multi-interface solution is not possible. It is clearly not possible to investigate all formal multi-interface solutions as the number of interface paths increases indefinitely (here an odd number), and in fact for three interfaces there are $6^3 = 216$ such solutions, all considerably more complex than the single-interface solutions. But given a single-interface solution with valid stress distributions over some finite region, it seems reasonable to assume that a multi-interface solution is not possible, since this requires further changes of state across each interface with no apparent mechanism available. When there is no valid stress distribution on one side of a single-interface, for any type, then further changes of state, with corresponding interfaces, are clearly required.

The validity conditions for the solutions of types B, C, D, E and F are most conveniently summarized in the two separate cases $F' \neq G'$ and $F' = G'$, where all functions are evaluated at the initial point O for application to some initial time interval. Conditions necessary for the validity of each type of solution in these two cases are listed in table 1 and 2 respectively, and with the exception of those marked*, are also sufficient to ensure validity. Various alternatives apply for different relative values of M, N, S. A valid solution of type A requires the additional initial (incident) wave function E, not identically zero, and so is not applicable initially to the meeting interaction.

Recalling the definitions (5·1·13) and (5·3·18), (5·4·13) it follows that $F^{(M)}0 > 0$, $(-1)^{M-1}G^{(M)} > 0$ and $\mathscr{C}\langle M\rangle < 0 \Rightarrow \mathscr{B}\langle M\rangle < 0$, and that $(-1)^N G^{(N)} > 0$, $(-1)^N Y^{(N)} > 0$ and $\mathscr{D}\langle N\rangle > 0 \Rightarrow \mathscr{E}\langle N\rangle > 0$. With these results, which distinguish between certain of the validity conditions for types D and F, it is evident from tables 1 and 2 that the sets of conditions for the five different types of solution are mutually exclusive. In the case $F' \neq G'$, table 1, it also follows that the different sets include the majority of possible initial conditions, although the sets marked* are not proved sufficient. The range of conditions not included is for $2G' = Y'$ with $S < N$ and $(-1)^{S-1}\,Y^{(S)} > 0$, or $S = N$, $(-1)^{N-1}\,Y^{(N)} > 0$ and $\mathscr{E}\langle N\rangle < 0$. Thus the existence of a single-interface solution is demonstrated over a wide range of conditions. In the case $F' = G'$, table 2, there is the obvious range of initial conditions $F^{(M)} < 0$ and $(-1)^{N-1}\,G^{(N)} < 0$ or $(-1)^{N-1}\,G^{(N)} > 0$, $M \leqslant N$, for which no single-interface solution exists.

Table 1. Necessary conditions for validity of meeting interaction solutions, $F' \neq G'$; * denotes conditions not sufficient to ensure validity

$0 \leqslant F' < G'$	B
$2F' > 2G' = Y'$	D $S > N$, $(-1)^{N-1}G^{(N)} > 0$ $S < N$, $(-1)^{S}\,Y^{(S)} > 0$ $S = N$, $(-1)^{N-1}G^{(N)} > 0$, $(-1)^{N}\,Y^{(N)} > 0$, or $(-1)^{N-1}G^{(N)} > 0$, $(-1)^{N-1}\,Y^{(N)} > 0$ and $\mathscr{E}\langle N\rangle > 0$, or $(-1)^{N}G^{(N)} > 0$, $(-1)^{N}\,Y^{(N)} > 0$ and $\mathscr{D}\langle N\rangle > 0$* E $S = N$, $(-1)^{N}G^{(N)} > 0$, $(-1)^{N}\,Y^{(N)} > 0$ and $\mathscr{D}\langle N\rangle < 0$, $\mathscr{E}\langle N\rangle > 0$ F $(-1)^{N}G^{(N)} > 0$* and $S > N$, $S < N$, $(-1)^{S-1}\,Y^{(S)} > 0$ $S = N$, $(-1)^{N-1}\,Y^{(N)} > 0$, or $(-1)^{N}\,Y^{(N)} > 0$ and $\mathscr{E}\langle N\rangle < 0$
$2F' > 2G' > Y'$	D
$F' > G'$, $2G' < Y' \leqslant 0$	F

For brevity, the conditions listed in tables 1 and 2 assume a definite sign for $\mathscr{B}\langle M\rangle$, $\mathscr{C}\langle M\rangle$, $\mathscr{D}\langle N\rangle$ and $\mathscr{E}\langle N\rangle$. It has been shown that if one of these expressions vanishes, then the first corresponding non-vanishing expression in the higher derivatives must take the appropriate sign, and this extension is a supplement to the listed conditions. The special linear profile case $F' = G' > 0$, $F^{(M)} = G^{(M)} = 0$ $(M \geqslant 2)$, when table 2 is not applicable, is treated in appendix B, where it is shown that solutions of types C, B, and D are all equivalent. Type F is excluded by $G' > 0$, and also type E since $Y' \leqslant 0$ initially is not permitted. That is, the stress distributions are identical and an interface is not strictly defined, being replaced by a finite region of uniform stress. Further, the cases $\mathscr{B}\langle M\rangle = 0$, $\mathscr{C}\langle M\rangle = 0$ $(M \leqslant 2)$, $\mathscr{D}\langle N\rangle = 0$, $\mathscr{E}\langle N\rangle = 0$ $(N \geqslant 2)$, when there is no first corresponding non-vanishing expression, have been shown to apply to one type of solution only, ensuring uniqueness.

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With the exception of the cases $F' < G'$ and $2F' > 2G' > Y'$, the criteria distinguishing between the different types of solution depend on higher derivatives of the initial wave functions and yield stress distribution. Such sensitive distinctions, required with an appropriate equality between first derivatives of the initial functions, would not be noted in trial numerical solutions, and an invalid type of solution may be accepted. The difference in

Table 2. Necessary conditions for validity of meeting interaction solutions, $F' = G'$; * denotes conditions not sufficient to ensure validity

$F' = G'$	$(-1)^{N-1}G^{(N)} > 0$	$(-1)^{N-1}G^{(N)} < 0$
$F^{(M)} > 0$	B $M > N$, $M = N$, $\mathscr{B}\langle M\rangle > 0$ C $M = N$, $\mathscr{B}\langle M\rangle < 0 < \mathscr{C}\langle M\rangle$ D $M < N$, $M = N$, $\mathscr{C}\langle M\rangle < 0$* and $2G' > Y'$ $(\dot{X} = 0)$, or $2G' = Y'$ $S > N$, $S > N$, $(-1)^S Y^{(S)} > 0$ $S = N$, $(-1)^N Y^{(N)} > 0$, or $(-1)^{N-1} Y^{(N)} > 0$ and $\mathscr{E}\langle N\rangle > 0$	F $2G' = Y' = 0$ $S = N$, $(-1)^N Y^{(N)} > 0$ and $\mathscr{E}\langle N\rangle < 0$ E $2G' = Y'$ $S = N$, $(-1)^N Y^{(N)} > 0$ and $\mathscr{D}\langle N\rangle < 0$, $\mathscr{E}\langle N\rangle > 0$ D $2G' > Y'(\dot{X} = 0)$, or $2G' = Y'(\dot{X} \neq 0)$ $S < N$, $(-1)^S Y^{(S)} > 0$ $S = N$, $(-1)^N Y^{(N)} > 0$ and $\mathscr{D}\langle N\rangle > 0$*
$F^{(M)} < 0$	B $M < N(F' > 0)$	

local stress from the actual valid distribution will be small, but the incorrect type of solution can predict a significantly different interface motion, which in turn influences the future progress of the interaction. It is not clear whether the resulting stress distribution at later times, having passed over this small local error, would be significantly changed.

The above criteria for the validity of solutions of types B to F in a subsequent time interval may be applied at any point on the interface path, once the previous valid solution is determined. That is, the valid solution over a long time may be constructed as a series of short-time solutions, applying the criteria successively at each new initial interface point. Of course the above conditions relate only to single oncoming elastic and plastic waves, $G(x-c_0t)$ and $F(x+c_1t)$, incident on the interface, while at a general interaction point, one

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of the other waves $H(x+c_0t)$, $E(x-c_1t)$, may be incident. Such waves are generated by the earlier interaction, and one will be currently incident if the interface velocity lies in the appropriate range, as discussed in §3. Thus the progress of the interaction, and in particular uniqueness of solution at each stage, requires an investigation of all possible types of solution under general 'initial' conditions. This is completed in the next section, where six sets of mutually exclusive conditions for the six types are derived.

While the validity criteria for the current type of solution continue to hold at each point on the interface path, then no change of type is possible (by the mutually exclusive uniqueness property), and a long time solution may be described by a single type. Furthermore, a change of type (or possible breakdown of all single-interface solutions) is required as soon as the criteria are violated at the current interface point, and the new type (if any) is uniquely determined by the current conditions. At such change-overs the interface velocity may pass continuously between the two ranges, or change discontinuously. In particular, the latter situation occurs when the governing criteria change abruptly due to discontinuities in the relevant derivatives, that is, the left- and right-hand derivatives of a given function are not equal. The subsequent solution is governed by the appropriate left- or right-hand derivatives of the different functions, and such discontinuities can change the effect of the respective stress variation, since the different criteria, through the derivatives, are statements of the required relative steepness of the wave profiles and yield stress distribution. Furthermore, in this sense, it is possible to interpret some of the validity criteria without the existence of left- or right-hand derivatives.

6. General interaction

The most general initial conditions involve waves propagating in the positive and negative directions in both elastic and plastic regions, and therefore include prescribed wave functions $H(x+c_0t)$ and $E(x-c_1t)$ in addition to the meeting wave functions $G(x-c_0t)$ and $F(x+c_1t)$ already treated. Waves $H(x+c_0t)$ or $E(x-c_1t)$ incident on the current interface can arise from previous stages of the interaction, or from separate interactions, and, together with $G(x-c_0t)$, $F(x+c_1t)$, $Y(x)$, are restricted only in the sense that the new initial stress distributions are valid. An interaction solution over a finite period can involve repeated changes in type of solution, and at each stage the new initial conditions of general form must be considered. General solutions for each of the six types of interface will now be determined, together with conditions for their validity which show that at most one type of single-interface solution is valid for given initial conditions. The existence of valid solutions of the super-fast type A is demonstrated. Again, for various conditions no single-interface solution is valid.

Reference to figures 5, 7 and 8 shows that in the interactions of types C, D, and E, which cover the range of interface velocity $-c_0 < \dot{X} < c_1$, neither of the initial waves $H(x+c_0t)$, $E(x-c_1t)$ would be incident on the interface. Such waves are represented by negative characteristics below ON, and positive characteristics below OZ, respectively, and do not intersect the interface path OP. Thus the solutions and validity conditions for types C, D and E derived in §§4 and 5 are unchanged by the additional initial waves. Recall that general forms of $G(x)$, $Y(x)$, not valid for the single elastic wave were included. However, reference to figures 6 and 9 show that an initial wave $E(x-c_1t)$ would be incident on an interface of type B, and an initial wave $H(x+c_0t)$ would be incident on an interface of type F. It therefore remains to consider interaction solutions of types B, A and F under the general initial conditions. The previous validity conditions for types B and F are now modified.

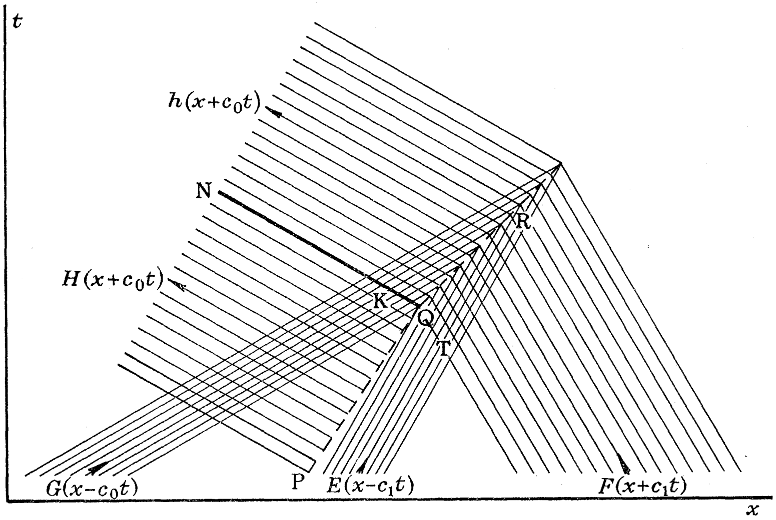

FIGURE 10. General characteristics diagram for fast plastic-elastic interface, type B; interface paths shown – – –.

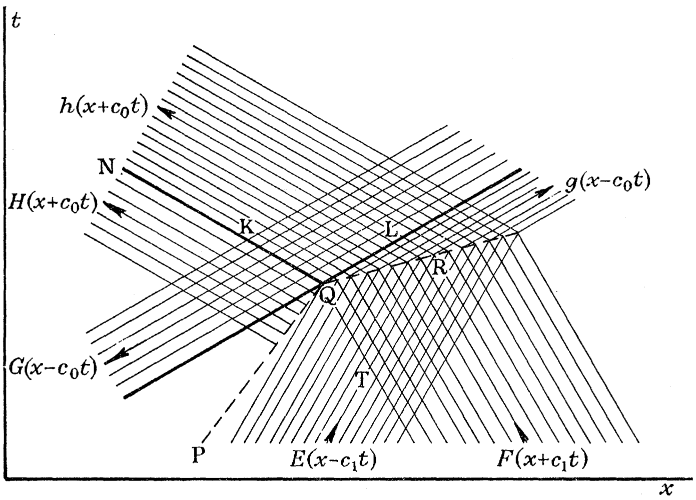

FIGURE 11. General characteristics diagram for super-fast plastic-elastic interface, type A; interface paths shown – – –.

In figures 10, 11 and 12, Q represents the initial point of the interface path QR for the appropriate interaction, while the path PQ is included to indicate an undefined previous interaction stage and is arbitrarily positioned. Clearly PQ and QR may both be paths of the same type, that is, no change of type occurs at Q, when the additional incident wave $H(x+c_0t)$ or $E(x-c_1t)$ is due to alternative or earlier interactions. Thus the validity conditions allow an interaction solution to be constructed step-by-step, either involving change

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of type or remaining within the same type. Each type of solution is examined over a subsequent time interval and the validity conditions are expressed in terms of functions evaluated at Q. In particular, $F^{(M)}$, $G^{(N)}$, $Y^{(S)}$ will now refer to the appropriate wave function and yield stress distribution derivatives evaluated at Q. Further, ignoring the first derivatives H', E' as before, define $H^{(P)}$, $E^{(L)}$ to be the first non-vanishing derivatives ($P, L \geqslant 2$) of H and E at Q.

Examination of the general interaction solutions follows closely the lines of the meeting interaction treatment, and less discussion and detail will be given.

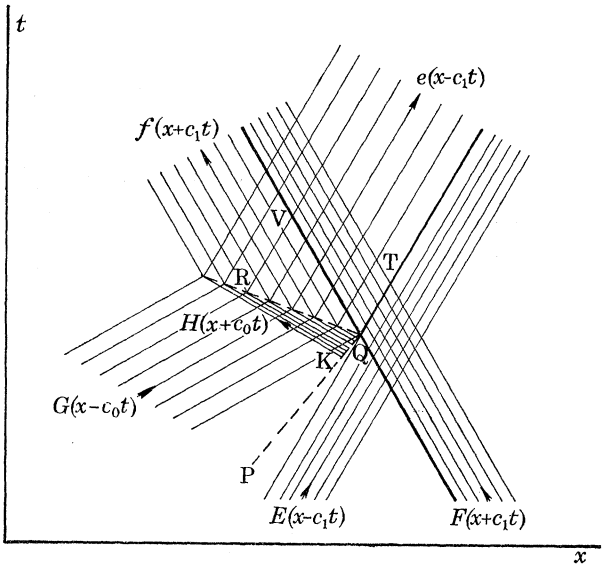

FIGURE 12. General characteristics diagram for super-fast elastic-plastic interface, type F; interface paths shown ---.

6·1. *Solution for a fast plastic-elastic interface, type B*

The characteristics diagram is shown in figure 10 with the type B interface represented by QR, and the interface velocity satisfies $c_0 \geqslant \dot{X} \geqslant c_1$. The additional initial wave $E(x-c_1 t)$ is incident on the interface and modifies the type B solution derived in §4.2. Initial and stress continuity conditions again give (4·1·2) to (4·1·4) if e is replaced by E, and O by Q, and similarly particle velocity continuity gives (4·1·11) which becomes an implicit equation for $X(t)$, namely

$$2c_1\{Y_Q - G[X(t)-c_0 t]\} - (c_0-c_1)\{F[X(t)+c_1 t] - Y_Q\} + (c_0+c_1)\,E[X(t)-c_1 t] = 0. \quad (6·1·1)$$

Here we have set $E_Q = H_Q = h_Q = 0$ for convenience, when $G_Q = F_Q = Y_Q$ and (6·1·1) is automatically satisfied at Q. Differentiating (6·1·1) gives the two interface velocity results

$$(\dot{X}-c_1)\{2c_1 G' + (c_0-c_1)F' - (c_0+c_1)E'\} = 2c_1(c_0-c_1)(G'-F'), \quad (6·1·2)$$

$$(c_0-\dot{X})\{2c_1 G' + (c_0-c_1)F' - (c_0+c_1)E'\} = (c_0^2-c_1^2)(F'-E'). \quad (6·1·3)$$

Now using the validity requirements (3·3),

$$\frac{1}{c_0}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{e}} = \frac{2c_0(\dot{X}+c_1)}{(c_0+c_1)\,(\dot{X}+c_0)}\,(F'-G') \leqslant 0, \tag{6·1·4}$$

$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{p}} = F'-E' \geqslant 0, \tag{6·1·5}$$

so that

$$E' \leqslant F' \leqslant G'. \tag{6·1·6}$$

Thus from (6·1·2) and (6·1·3),

$$\dot{X} = c_1 \Rightarrow G' = F', \quad E' < F' = G' \Rightarrow \dot{X} = c_1, \tag{6·1·7}$$

$$\dot{X} = c_0 \Rightarrow F' = E', \quad F' = E' < G' \Rightarrow \dot{X} = c_0. \tag{6·1·8}$$

If $E' < F' < G'$ at Q, then the elastic and plastic validity conditions are satisfied in some neighbourhood of Q for $x \geqslant x_{\mathrm{Q}}$ beyond the first incident characteristic TQ, and also in some neighbourhood $x < x_{\mathrm{Q}}$ since it follows from $F'_{\mathrm{Q}} < G'_{\mathrm{Q}}$ that $Y_{\mathrm{K}} - \sigma_{\mathrm{K}} > 0$ as K→Q. Validity below QK is assumed satisfied, being a condition on the initial wave function H not involved in the interaction. Further, $\dot{X}$ lies strictly within the required range.

When $F' = E'$ at Q, then plastic validity requires that $F'-E'$ becomes positive immediately beyond Q, and that the stress is loading at T as T→Q. Now

$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{T}} = E'_{\mathrm{Q}} - E'_{\mathrm{R}} > 0, \tag{6·1·9}$$

and hence

$$F' = E' \Rightarrow E^{(L)} < 0. \tag{6·1·10}$$

The former requirement is expressed by differentiating $F'-E'$ until a non-vanishing expression is obtained, thus

$$(\dot{X}+c_1)^{m-1}\,F^{(m)} - (\dot{X}-c_1)\,E^{(m)} > 0. \tag{6·1·11}$$

If $L < M$, that is $E^{(L)} \neq 0$, $F^{(L)} = 0$, and $\dot{X} \neq c_1$, (6·1·11) is satisfied in view of (6·1·10), while if $L \leqslant M$ and $\dot{X} = c_1$, only possible if $F' = G'$ by (6·1·8), or if $L > M$, the required condition is $F^{(M)} > 0$. That is

$$L \leqslant M, \quad \dot{X} = c_1, \quad \text{or} \quad L > M \Rightarrow F^{(M)} > 0. \tag{6·1·12}$$

For $L = M$, (6·1·11) is trivially satisfied if $F^{(M)} > 0$, noting (6·1·10), while if $F^{(M)} < 0$ and $\dot{X} \neq c_1$, it requires

$$\frac{-F^{(M)}}{-E^{(M)}} < \left(\frac{\dot{X}-c_1}{\dot{X}+c_1}\right)^{M-1} < \left(\frac{c_0-c_1}{c_0+c_1}\right)^{M-1}. \tag{6·1·13}$$

The latter inequality is necessary, and is sufficient only if $\dot{X} = c_0$, which includes, by (6·1·8), the case $F' < G'$. Define

$$\mathscr{A}\langle m\rangle = (c_0-c_1)^{m-1}\,E^{(m)} - (c_0+c_1)^{m-1}\,F^{(m)}, \tag{6·1·14}$$

then (6·1·13) becomes

$$F^{(M)} < 0, \quad \mathscr{A}\langle M\rangle < 0. \tag{6·1·15}$$

If $F' = G'$ at Q, then elastic validity requires that $F'-G'$ decreases, and that $Y_{\mathrm{K}} - \sigma_{\mathrm{K}} > 0$ as K→Q, now equivalent to $Y_{\mathrm{K}} - G_{\mathrm{K}} > 0$. From an expansion in $t_{\mathrm{K}} - t_{\mathrm{Q}}$ it is found that the latter condition requires

$$2G' \geqslant Y', \tag{6·1·16}$$

and, if $2G' = Y'$, the additional implications

$$\left.\begin{aligned} &S < N \Rightarrow (-1)^S Y^{(S)} > 0, \quad S > N \Rightarrow (-1)^{N-1} G^{(N)} > 0, \\ &S = N \Rightarrow \mathscr{E}\langle N\rangle > 0, \end{aligned}\right\} \qquad (6\cdot1\cdot17)$$

recalling the definition (5·4·13). The expression analogous to (6·1·11) for the change of $F' - G'$ leads to the following requirements:

$$\left.\begin{aligned} &M < N \quad \text{or} \quad M \geqslant N, \quad \dot{X} = c_0 \Rightarrow F^{(M)} < 0, \\ &M > N, \quad \dot{X} \neq c_0 \Rightarrow (-1)^{N-1} G^{(N)} > 0. \end{aligned}\right\} \qquad (6\cdot1\cdot18)$$

If $M = N$, then necessarily

$$F^{(M)} > 0 \Rightarrow (-1)^{M-1} G^{(M)} > 0, \quad \mathscr{B}\langle M\rangle > 0, \qquad (6\cdot1\cdot19)$$

recalling the definition (5·1·13), where the latter inequality is also sufficient if $\dot{X} = c_1$, while the case $F^{(M)} < 0$, $(-1)^{M-1} G^{(M)} > 0$ is trivially valid. Further, any values of $F^{(M)} < 0$, $(-1)^{M-1} G^{(M)} < 0$ may be valid, depending on $\dot{X}$, but if $\dot{X} = c_1$ the condition is that $\mathscr{B}\langle M\rangle > 0$.

When $E' = F' = G'$ at Q, no explicit expression for $\dot{X}$ is given by (6·1·1), and in particular it is not necessarily an end-point velocity c_1 or c_0, so that sufficiency of some of the previous validity conditions cannot be affirmed. Continued differentiation of (6·1·1) in the case $E' = F' = G' = 0$ leads to various conditions under which $\dot{X}$ does take an end-point value. In both situations, if $\mathscr{A}\langle M\rangle = 0$ is accompanied by $\dot{X} = c_0$, the requirement that the first non-vanishing $\mathscr{A}\langle M\rangle < 0$ is equivalent to $\dot{X}$ decreasing, and similarly if $\mathscr{B}\langle M\rangle = 0$ and $\dot{X} = c_1$, then $\mathscr{B}\langle m\rangle > 0$ is equivalent to $\dot{X}$ increasing; see appendix C.

6·2. *Solution for a super-fast plastic-elastic interface, type A*

The characteristics diagram is shown in figure 11 with the type A interface represented by QR, and the interface velocity satisfies $c_0 < \dot{X}$. Again the initial elastic wave $H(x + c_1 t)$ does not influence the interaction, except that a valid elastic stress distribution is assumed in some neighbourhood of Q up to the first reflected characteristic QK. Further, the initial wave $G(x - c_0 t)$ is defined only in the domain beyond QL, and does not enter the solution for $X(t)$ and the subsequently formed waves $h(x + c_0 t)$ and $g(x - c_0 t)$, but is restricted for validity in some neighbourhood KQL. The elastic wave $g(x - c_0 t)$ is continuously created by the interaction, but follows the interface at slower speed. Initial and continuity conditions lead to (4·1·2) to (4·1·4) and (4·1·10), (4·1·11) with the replacements $G \to g$, $e \to E$, $\mathrm{O} \to \mathrm{Q}$, and the further interface condition (3·7) provides an implicit equation for $X(t)$, namely

$$F'[X(t) + c_1 t] = E'[X(t) - c_1 t]. \qquad (6\cdot2\cdot1)$$

The restriction $F'_Q = E'_Q$ is necessary for a type A solution, which will therefore not commonly occur in practice. From (4·1·10), (4·1·11) and (6·2·1), it also follows that

$$h'[X(t) + c_0 t] = g'[X(t) - c_0 t] = F'[X(t) + c_1 t] = E'[X(t) - c_1 t]. \qquad (6\cdot2\cdot2)$$

Continued differentiation of (6·2·1) shows, since $\dot{X} > c_0$, that $L = M$ and

$$(\dot{X} + c_1)^{M-1} F^{(M)} = (\dot{X} - c_1)^{M-1} E^{(M)}, \qquad (6\cdot2\cdot3)$$

so that $F^{(M)}$, $E^{(M)}$ have the same sign. Since $\partial\sigma/\partial t$ vanishes on QR by (3·7), plastic validity requires $\partial\sigma/\partial t > 0$ as $\mathrm{T}\to\mathrm{Q}$. Now

$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{T}} = F'_{\mathrm{Q}}-F'_{\mathrm{R}} = E'_{\mathrm{Q}}-E'_{\mathrm{R}}, \quad < 0 \tag{6·2·4}$$

implies
$$F^{(M)} < 0, \quad E^{(M)} < 0. \tag{6·2·5}$$

Hence, from (6·2·3),
$$\left(\frac{c_0-c_1}{c_0+c_1}\right)^{M-1} < \frac{-F^{(M)}}{-E^{(M)}} < 1, \tag{6·2·6}$$

where the left-hand equality is the limit as $\dot{X}\to c_0$, and the right-hand equality is the limit as $\dot{X}\to\infty$. Recalling the definition (6·1·14), (6·2·6) requires

$$\mathscr{A}\langle M\rangle > 0, \tag{6·2·7}$$

and if $\mathscr{A}\langle M\rangle = 0$ then the first non-vanishing $\mathscr{A}\langle m\rangle > 0$, which is equivalent to $\dot{X}$ increasing from c_0; see appendix C. Note that (6·2·1) and (6·2·3) imply that $E(x)$ is not identically zero, verifying that type A is not a possible solution to the meeting interaction.

Again by (3·7), elastic validity in a neighbourhood RQL requires that $(\partial\sigma/\partial t)_{\mathrm{L}} < 0$ as $\mathrm{L}\to\mathrm{Q}$. Now

$$\frac{1}{c_0}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{L}} = -g'_{\mathrm{Q}}+g'_{\mathrm{R}} < 0 \tag{6·2·8}$$

implies that the first non-vanishing $g^{(m)} < 0$ $(m > 1)$, and by (6·2·2), since $\dot{X} > c_0$, this implies $m = M$ and $F^{(M)} < 0$, guaranteed by (6·2·5). For validity in KQL, the requirement in $x < x_{\mathrm{Q}}$ is that $Y_{\mathrm{K}}-\sigma_{\mathrm{K}} > 0$ as $\mathrm{K}\to\mathrm{Q}$, which is just a repeat of the same type B situation, and again implies (6·1·16) and (6·1·17). At Q,

$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{Q}} = -G'_{\mathrm{Q}}+F'_{\mathrm{Q}} < 0, \tag{6·2·9}$$

which requires $F'_{\mathrm{Q}} < G'_{\mathrm{Q}}$, and further, if $F'_{\mathrm{Q}} = G'_{\mathrm{Q}}$, that

$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{L}} = -G'_{\mathrm{Q}}+F'_{\mathrm{R}} = -F'_{\mathrm{Q}}+F'_{\mathrm{R}} < 0, \tag{6·2·10}$$

as $\mathrm{L}\to\mathrm{Q}$. Thus
$$F' < G' \quad \text{or} \quad F' = G' \quad \text{and} \quad F^{(M)} < 0. \tag{6·2·11}$$

6·3. *Solution for a super-fast elastic-plastic interface, type F*

The characteristics diagram is shown in figure 12 with the type F interface represented by QR, and the interface velocity satisfies $\dot{X} \leqslant -c_0$. The initial plastic waves $E(x-c_1t)$, $F(x+c_1t)$ do not influence the interaction solution, but are relevant to a valid distribution in a neighbourhood TQV. A plastic wave $f(x-c_1t)$, created by the interaction, follows the interface at slower speed. Both initial elastic waves $G(x-c_0t)$ and $H(x+c_0t)$ are incident on the interface. Again (4·1·1) to (4·1·4) and (4·1·10), (4·1·11) apply with the replacements $\mathrm{O}\to\mathrm{Q}$, $h\to H$, $F\to f$, and the further interface condition (3·12) provides an implicit equation for $X(t)$, namely

$$G[X(t)-c_0t]+H[X(t)+c_0t] = Y[X(t)], \tag{6·3·1}$$

automatically satisfied at Q. Differentiating gives the interface velocity expressions

$$\dot{X}\{G'+H'-Y'\} = -c_0(H'-G'), \tag{6·3·2}$$

$$(\dot{X}+c_0)\{G'+H'-Y'\} = -c_0(Y'-2G'). \tag{6·3·3}$$

Validity requires that the stress is loading on both elastic and plastic sides of the interface, by (3·12), so that on using (4·1·10) and (4·1·11),

$$\left.\begin{aligned}\frac{1}{c_0}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{e}} &= -G'+H' \geqslant 0,\\ \frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{p}} = f'-e' &= \frac{c_1(c_0-\dot{X})\,(c_0+\dot{X})}{c_0(c_1-\dot{X})\,(c_1+\dot{X})}\,(H'-G') \geqslant 0,\end{aligned}\right\} \tag{6·3·4}$$

and hence $$H' \geqslant G'. \tag{6·3·5}$$

Further, from (4·1·10) and (4·1·11), if $H' = G'$ at Q, then

$$H' = G' = e' = f', \tag{6·3·6}$$

while if $\dot{X} = -c_0$, $$G' = e' = f'. \tag{6·3·7}$$

Now since $\dot{X} \leqslant -c_0$, it follows from (6·3·2) and (6·3·3) that

$$\left.\begin{aligned}\dot{X} = -c_0 \quad &\text{or} \quad H' = G' \Rightarrow 2G' = Y',\\ \dot{X} \neq -c_0 \quad &\text{and} \quad 2G' = Y' \Rightarrow H' = G',\\ H' > G' &\Rightarrow 2G' \leqslant Y',\end{aligned}\right\} \tag{6·3·8}$$

so in view of (6·3·5), always $$2G' \leqslant Y'. \tag{6·3·9}$$

If, at Q, $Y' > 2G'$, then noting (6·3·8), $H' > G'$ and $\dot{X} \neq -c_0$, and both strict inequalities of (6·3·4) hold. With $H' > G'$ and $2G' = Y'$, the strict elastic inequality holds, but the plastic inequality also requires $\dot{X} \neq -c_0$, and if $\dot{X} = -c_0$ it is necessary that $Y'-2G'$ increase along the path so that $\dot{X}$ decreases, as required anyway by (6·3·9). Further, if $H'-G' = 0$, when $Y'-2G' = 0$, both quantities must increase. These conditions will ensure elastic validity near Q, subject to an initially valid distribution. It remains to consider cases in which at least one of the conditions $H' = G'$ or $\dot{X} = -c_0$ holds and $(\partial\sigma/\partial t)_{\mathrm{p}}$ vanishes at Q.

First note the result that at each interface point

$$e' \geqslant G', \tag{6·3·10}$$

which follows from (4·1·10), (4·1·11) and (6·3·5), where the equality holds if, and only if, $H' = G'$ and $\dot{X} = -c_0$ $(2G' = Y')$. As T→Q and V→Q along the respective limit characteristics of the F, e domain, plastic validity requires

$$\left.\begin{aligned}\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{T}} &= F'_{\mathrm{T}}-e'_{\mathrm{Q}} > 0,\\ \frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{V}} &= F'_{\mathrm{Q}}-e'_{\mathrm{R}} > 0.\end{aligned}\right\} \tag{6·3·11}$$

From the first and (6·3·10) it is necessary that $F'_{\mathrm{Q}} \geqslant G'_{\mathrm{Q}}$, and from the second that $F'_{\mathrm{Q}} \geqslant G'_{\mathrm{R}}$ and hence, if $F'_{\mathrm{Q}} = G'_{\mathrm{Q}}$, then $(-1)^N G^{(N)} > 0$. Further, as V→Q in the f, e domain,

$$\frac{1}{c_1}\left(\frac{\partial\sigma}{\partial t}\right)_{\mathrm{V}} = f'_{\mathrm{Q}}-e'_{\mathrm{R}} > 0, \tag{6·3·12}$$

and since $H' = G'$ or $\dot{X} = -c_0$ at Q, (6·3·6), and (6·3·7) show that $f'_{\mathrm{Q}} = G'_{\mathrm{Q}}$, so that a necessary condition is that $G'_{\mathrm{Q}} > e'_{\mathrm{R}} \geqslant G'_{\mathrm{R}}$; thus

$$(-1)^N G^{(N)} > 0. \tag{6·3·13}$$

Note that $2G' = Y' \Rightarrow H' = G'$ or $\dot{X} = -c_0$ by (6·3·8), and hence implies (6·3·13). When neither $H' = G'$ nor $\dot{X} = -c_0$, it has been seen that the stress is strictly loading in some neighbourhood VQR. Also, letting T→Q in the F, E domain, it follows that

$$F' > E' \quad \text{or} \quad F' = E' \quad \text{and} \quad F^{(M)} > 0. \tag{6·3·14}$$

Returning to the case $Y' - 2G' = 0$ at Q, and so required to increase along the interface path, the results (5·5·13), (5·5·14) again follow, noting that (6·3·13) holds. Further, the condition (5·5·15) is necessary, but only sufficient if $\dot{X} = -c_0$. Again, if $\mathscr{E}\langle N\rangle = 0$ is accompanied by $\dot{X} = -c_0$, then $\dot{X}$ decreasing and the first non-vanishing $\mathscr{E}\langle m\rangle < 0$ are equivalent; see appendix C. Finally, if $H' - G' = 0$ at Q, the increasing requirement becomes (6·3·13) again if $\dot{X} = -c_0$ or $P > N$, together with the alternatives

$$P < N, \quad \dot{X} \neq -c_0 \Rightarrow (-1)^{P-1} H^{(P)} > 0, \tag{6·3·15}$$

$$P = N \Rightarrow (-1)^{N-1} H^{(N)} > 0 \quad \text{or} \quad \frac{(-1)^N G^{(N)}}{(-1)^N H^{(N)}} > \left(\frac{-c_0 - \dot{X}}{c_0 - \dot{X}}\right)^{N-1} > 0, \tag{6·3·16}$$

where the latter inequality provides no explicit condition.

6·4. *Validity conditions for the six types of solution, uniqueness*

Validity of the initial stress distributions has been tacitly assumed in the general solutions, that is, in some neighbourhood of Q within the domains governed by characteristics reflected from the interface path up to the point Q. In fact this implies conditions on the wave functions $H(x + c_0 t)$, $E(x - c_1 t)$ and yield stress distribution $Y(x)$, evaluated at Q. Where any of these functions are involved in the interaction these conditions are included in the general validity conditions, but $H(x + c_0 t)$ is not involved in types A to E, and $E(x - c_1 t)$ is not involved in the types C to F. Conditions for these two classes should then be supplemented by the additional initial distribution requirements. Complete conditions were derived for the meeting interaction solutions, but there $H(x + c_0 t)$, $E(x - c_1 t)$ were absent.

The additional conditions in the first class, types A to E, follow from the requirement that $Y_K - \sigma_K > 0$ as K→Q along the G-characteristic incident at Q, where a typical point K is that shown in figure 12. This is equivalent to $Y_K - G_Q - H_K > 0$, which may be expressed in terms of derivatives by an expansion about Q, recalling that $Y_Q - G_Q - H_Q = 0$, and the results are given in table 3. Similarly, in the second class, types C to F, it is required that $(\partial\sigma/\partial t)_T > 0$ as T→Q along the F-characteristic incident at Q, where a typical point T is that shown in figure 10. Thus $F'_Q - E'_T > 0$ is required, and the consequent results are given in table 3.

A full set of conditions necessary for validity of the general solutions of types B, A, and F are presented in the four tables 4 to 7. These specifically distinguish between the various cases of $F' \neq G'$ and $F' = G'$ for easier comparison with tables 1 and 2 where the conditions for types C, D and E are already general. Careful examination of the numerous subcases shows that the six sets of conditions for the six types are mutually exclusive, so verifying uniqueness of the single-interface solution for general initial conditions. It will be noted that many more conditions are not sufficient to ensure validity, those denoted by *. Each such condition is associated with a type B, D or F solution for which the interface path equation

Table 3. Supplementary initial validity conditions

A, B, C, D, E	$2H' > Y'$ or $2H' = Y'$ $S > P$, $(-1)^{P-1}H^{(P)} > 0$ $S < P$, $(-1)^{S}Y^{(S)} > 0$ $S = P$, $(-1)^{S}Y^{(S)} > 2^{S}(-1)^{S}H^{(S)}$
C, D, E, F	$F' > E'$ or $F' = E'$ and $E^{(L)} < 0$

Table 4. Necessary conditions for validity of general solutions $F' > G'$, * denotes conditions not sufficient to ensure validity

	$F' > G'$	
	F	
$2G' < Y'$	and	$G' < H'$
or $2G' = Y'$ and $(-1)^{N}G^{(N)} > 0$		or $G' = H'$ and $(-1)^{N}G^{(N)} > 0$
$S > N$,		$P < N$, $\dot{X} \neq -c_0$, $(-1)^{P-1}H^{(P)} > 0$
$S < N$, $(-1)^{S-1}Y^{(S)} > 0$		$P = N$*
$S = N$, $(-1)^{N}Y^{(N)} > 0$ and $\mathscr{E}\langle N\rangle < 0$*	and	
$E' < F'$	or	$E' = F'$, $F^{(M)} > 0$

Table 5. Necessary conditions for validity of general solutions, $F' < G'$; * denotes conditions not sufficient to ensure validity

$E' < F' < G'$	B
$E' = F' < G'$	B $L > M$, $E^{(L)} < 0$, $F^{(M)} > 0$* $L \leqslant M$, $\dot{X} = c_1$, $L = M$, $\dot{X} \neq c_1$, $E^{(M)} < 0$, $F^{(M)} < 0$ and $\mathscr{A}\langle M\rangle < 0$*, and $2G' > Y'$ or $2G' = Y'$ $S > N$, $(-1)^{N-1}G^{(N)} > 0$ $S < N$, $(-1)^{S}Y^{(S)} > 0$ $S = N$, $\mathscr{E}\langle N\rangle > 0$ A $L = M$, $E^{(M)} < 0$, $F^{(M)} < 0$ and $\mathscr{A}\langle M\rangle > 0$, and $2G' > Y'$ or $2G' = Y'$ $S > N$, $(-1)^{N-1}G^{(N)} > 0$ $S < N$, $(-1)^{S}Y^{(S)} > 0$ $S = N$, $\mathscr{E}\langle N\rangle > 0$

TABLE 6. NECESSARY CONDITIONS FOR VALIDITY OF GENERAL SOLUTIONS, $E' < F' = G'$; * DENOTES CONDITIONS NOT SUFFICIENT TO ENSURE VALIDITY

$E' < F' = G'$	
B	*F*
$M < N$, $F^{(M)} < 0$	
$M \geqslant N$, $\dot{X} = c_0$,	$(-1)^N G^{(N)} > 0$
$M = N$, $\dot{X} \neq c_0$, $F^{(M)} > 0$, $(-1)^{M-1} G^{(M)} > 0$	and $2G' < Y'$
and $\mathscr{B}\langle M\rangle > 0$*,	or $2G' = Y'$
or $F^{(M)} < 0$, $(-1)^{M-1} G^{(M)} > 0$,	$S > N$
or $F^{(M)} < 0$, $(-1)^{M} G^{(M)} > 0$*	$S < N$, $(-1)^{S-1} Y^{(S)} > 0$
and $2G' > Y'$	$S = N$, $(-1)^N Y^{(N)} > 0$ and $\mathscr{E}\langle N\rangle < 0$*
or $2G' = Y'$	and $H' > G'$
$S > N$, $(-1)^{N-1} G^{(N)} > 0$	or $H' = G'$
$S < N$, $(-1)^{S} Y^{(S)} > 0$	$P < N$, $\dot{X} \neq -c_0$ and $(-1)^{P-1} H^{(P)} > 0$
$S = N$, $\mathscr{E}\langle N\rangle > 0$	$P = N$*

TABLE 7A. NECESSARY CONDITIONS FOR VALIDITY OF MEETING INTERACTION SOLUTIONS, $E' = F' = G'$; * DENOTES CONDITIONS NOT SUFFICIENT TO ENSURE VALIDITY

$E' = F' = G'$, $F^{(M)} > 0$	
$(-1)^{N-1} G^{(N)} > 0$	$(-1)^{N-1} G^{(N)} < 0$
B	*F*
$E^{(L)} < 0$	$2G' < Y'$,
	$G' < H'$
$M = N$ $\mathscr{B}\langle M\rangle > 0$*	or $2G' = Y'$
and $2G' > Y'$	$S > N$
or $2G' = Y'$	$S < N$, $(-1)^{S-1} Y^{(S)} > 0$
$S > N$	$S = N$, $(-1)^N Y^{(N)} > 0$ and $\mathscr{E}\langle N\rangle < 0$*,
$S < N$, $(-1)^S Y^{(S)} > 0$	and $G' < H'$
$S = N$, $\mathscr{E}\langle N\rangle > 0$	or $G' = H'$
	$P < N$, $\dot{X} \neq -c_0$ and $(-1)^{P-1} H^{(P)} > 0$
	$P = N$*

TABLE 7B. NECESSARY CONDITIONS FOR VALIDITY OF MEETING INTERACTION SOLUTIONS, $E' = F' = G'$; * DENOTES CONDITIONS NOT SUFFICIENT TO ENSURE VALIDITY

$E' = F' = G'$, $F^{(M)} < 0$			
$(-1)^{N-1}G^{(N)} > 0$		$(-1)^{N-1}G^{(N)} < 0$	
	B		B
$L = M$, $\dot{X} \neq c_1$,	$E^{(M)} < 0$ and $\mathscr{A}\langle M\rangle < 0$*,	$L = M$, $\dot{X} \neq c_1$,	$E^{(M)} < 0$ and $\mathscr{A}\langle M\rangle < 0$*
and	$M < N$ or $M \geqslant N$, $\dot{X} = c_0$	and	$M < N$ or $M \geqslant N$, $\dot{X} = c_0^*$
and	$2G' > Y'$	and	$2G' > Y'$
or	$2G' = Y'$	or	$2G' = Y'$
$S > N$		$S < N$,	$(-1)^S Y^{(S)} > 0$
$S < N$,	$(-1)^S Y^{(S)} > 0$	$S = N$,	$\mathscr{E}\langle N\rangle > 0$
$S = N$,	$\mathscr{E}\langle N\rangle > 0$		
	A		A
$L = M$,	$E^{(M)} < 0$ and $\mathscr{A}\langle M\rangle > 0$,	$L = M$,	$E^{(M)} < 0$ and $\mathscr{A}\langle M\rangle > 0$,
and	$2G' > Y'$	and	$2G' > Y'$
or	$2G' = Y'$	or	$2G' = Y'$
$S > N$			
$S < N$,	$(-1)^S Y^{(S)} > 0$	$S < N$,	$(-1)^S Y^{(S)} > 0$
$S = N$,	$\mathscr{E}\langle N\rangle > 0$	$S = N$,	$\mathscr{E}\langle N\rangle > 0$

involves three initial functions, and explicit results cannot always be deduced from the interface velocity inequalities. There are more situations in which no single-interface solution is valid, and a multi-interface solution is required.

Again definite signs are assumed for the expressions $\mathscr{A}\langle M\rangle$, $\mathscr{B}\langle M\rangle$ and $\mathscr{E}\langle N\rangle$ with the understanding that a vanishing expression requires the first corresponding non-vanishing expression to take the appropriate sign. If all the corresponding expressions vanish then the limit solution falls into one type, that with the closed interface velocity range. Special linear profile cases, in which two or three of the initial functions have identical constant first derivatives and all higher derivatives zero, arise as in the meeting interaction. The different types of solution for such cases are not distinguished by the results in the tables, but are equivalent in similar manner to the cases discussed in appendix B.

Finally, we may note that the second basic interaction involving only single elastic and plastic waves, an overtaking interaction, is described by $H \equiv 0$, $F \equiv 0$, $E' < 0$ (plastic loading), $G' > 0$ (elastic unloading) for which it is seen (table 4) that only a type B solution is possible, and in fact always exists. This was shown in [M].

CONCLUDING REMARKS

The uniqueness of the single-interface solution for general initial conditions has been proved, and the existence of such a solution shown for a wide range of conditions. We have therefore demonstrated, to a considerable extent, that the assumed elastic-plastic model provides a consistent description of the transient response to dynamic loading. Moreover, the validity conditions predetermine the type of solution to construct. However, it has not been proved that under conditions for which a single-interface solution exists, a multi-interface solution cannot exist also. Such a solution, though, requires further changes of state at the new interface, where in the single-interface solution no mechanism for this change exists. It appears then, very unlikely that a multi-interface solution will occur under these circumstances.

In cases where no single-interface solution exists, the sets of validity conditions which fail in each type of solution show, from their derivation, the region and cause of breakdown. That is, where the stress distributions are not consistent with the elastic-plastic model. This provides some indication of regions in which a new interface path develops, and perhaps of the type of interface required.

It has been pointed out that the choice of type of solution for given conditions may depend sensitively on higher derivatives of the initial functions, which a numerical finite-difference or characteristics scheme would not detect. An open question is whether local errors passed over by such a scheme build-up as the interaction proceeds, or in practice remain insignificant. However, typical initial conditions will not rely on such sensitive distinctions, and in fact the choice of solution will commonly be decided by inequalities between first derivatives of the initial functions. Furthermore, the solutions of types A, C and E, which require equalities between first derivatives, are likely to be uncommon, so that in most cases the choice is between types B, D and F. We recall here that the conditions imply the left or right-hand derivatives appropriate to the subsequent interval of time.

This investigation of uniqueness and existence of solutions to wave interactions arose from discussions held during a visiting appointment (L. W. M.) at the R.A.R.D.E., Fort Halstead, in 1963.

Notation

x is a Lagrangian coordinate along the axis of propagation.

t denotes time.

u, v are particle displacement and velocity respectively.

σ, ϵ are principal engineering compressive stress and strain components respectively in the x-direction.

c_0, c_1 are Lagrangian elastic and plastic wave speeds respectively.

$X(t)$ denotes current position of an interface between elastic and plastic regions.

$Y(x)$ is a yield stress distribution.

G, H and E, F denote initial elastic and plastic wave functions respectively, representing in order propagation in positive x- and negative x-directions.

L, M, N, P, S are orders of lowest non-vanishing derivatives (excluding first derivative) of the functions E, F, G, H, Y respectively.

$\mathscr{A}\langle m\rangle = (c_0-c_1)^{m-1}E^{(m)}-(c_0+c_1)^{m-1}F^{(m)}$.

$\mathscr{B}\langle m\rangle = (c_0-c_1)^{m-1}(-1)^{m-1}G^{(m)}-(2c_1)^{m-1}F^{(m)}$.

$\mathscr{C}\langle m\rangle = (2c_0)^{m-1}(-1)^{m-1}G^{(m)}-(2c_1)^{m-1}F^{(m)}$.

$\mathscr{D}\langle m\rangle = (c_1)^{m-1}(-1)^{m}Y^{(m)}-2(c_0+c_1)^{m-1}(-1)^{m}G^{(m)}$.

$\mathscr{E}\langle m\rangle = (-1)^{m}Y^{(m)}-2^{m}(-1)^{m}G^{(m)}$.

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Appendix A. Stress discontinuities at an interface

Since the initial stress wave distributions, and yield stress distribution, are continuous, any stress discontinuity which occurs subsequently must necessarily propagate along a characteristic emerging from the interface path, or along the interface path itself. The solutions based on stress continuity at the interface show that the stress waves generated at the interface are also continuous, so that it remains only to examine the possibility of a stress discontinuity at the interface. By (2·20) the interface speed is then either the elastic wave speed c_0, or the plastic wave speed c_1, with the choice governed by the instantaneous change of state. Four different cases must be considered:

(i) $\dot{X}(t) = c_0$, an elastic unloading discontinuity;

(ii) $\dot{X}(t) = c_1$, a plastic loading discontinuity;

(iii) $\dot{X}(t) = -c_1$, a plastic loading discontinuity;

(iv) $\dot{X}(t) = -c_0$, an elastic loading discontinuity.

In case (iii) the stress is at yield immediately ahead of the discontinuity, while in case (iv) the stress is at yield immediately behind the discontinuity. Analogous cases arise for the reverse yield situation.

It can be assumed that the stress discontinuity first forms at $t = 0$, and the initial continuous conditions are those for the general interaction described in §6. In fact for each of the above interface velocities an initial elastic wave $H(x+c_0 t)$ is not incident on the interface, but an initial plastic wave $E(x-c_1 t)$ does affect case (i). Further, an initial elastic wave $G(x-c_0 t)$ does not affect case (i), while an initial plastic wave $F(x+c_1 t)$ does not affect cases (iii) and (iv). The magnitude of the stress discontinuity is represented by $\Delta(t)$, where $\Delta(0) = 0$, and it is shown that in all four cases $\Delta(t) \equiv 0$ for $t \geqslant 0$; that is, no stress discontinuity can be formed at the interface by initially continuous stress waves. The formal solutions are determined by the initial conditions and the stress-particle velocity jump relation (2·20), together with the yield condition in cases (iii) and (iv).

(i) $X(t) = c_0 t \quad (t \geqslant 0)$,

$$F[(c_0+c_1)\,t] + E[(c_0-c_1)\,t] = \Delta(t) + \sigma_0 + h[2c_0 t], \tag{A 1}$$

$$c_0\{\sigma_0 - F[(c_0+c_1)\,t] + E[(c_0-c_1)\,t]\} = c_1\{\Delta(t) - h[2c_0 t]\}. \tag{A 2}$$

Eliminating $h[2c_0 t]$ gives

$$2c_1\Delta(t) = (c_0-c_1)\{\sigma_0 - F[(c_0+c_1)\,t]\} + (c_0+c_1)\,E[(c_0-c_1)\,t], \tag{A 3}$$

and hence
$$2c_1\,\Delta'(t) = (c_0^2-c_1^2)\,\{E'[(c_0-c_1)\,t]-F'[(c_0+c_1)\,t]\}. \tag{A 4}$$

But
$$(1/c_1)\,(\partial\sigma/\partial t)_{\mathrm{p}} = F'[(c_0+c_1)\,t]-E'[(c_0-c_1)\,t], \tag{A 5}$$

so that
$$2c_1^2\Delta'(t) = -(c_0^2-c_1^2)\,(\partial\sigma/\partial t)_{\mathrm{p}}. \tag{A 6}$$

Now validity of the plastic stress distribution requires $(\partial\sigma/\partial t)_{\mathrm{p}} \geqslant 0$ so that (A 6) implies $\Delta'(t) < 0$ for all $t \geqslant 0$, and hence $\Delta(t) \leqslant 0$ for all $t \geqslant 0$ since $\Delta(0) = 0$. In view of the definition $\Delta(t) \geqslant 0$ it follows that $\Delta(t) > 0$ cannot occur for any $t > 0$; that is, no discontinuity forms.

(ii) $X(t) = c_1 t \quad (t \geqslant 0)$,

$$G[-(c_0-c_1)\,t]+h[(c_0+c_1)\,t] = \Delta(t)+F[2c_1 t], \tag{A 7}$$

$$c_1\{G[-(c_0-c_1)\,t]-\sigma_0-h[(c_0+c_1)\,t]\} = c_0\{\Delta(t)+\sigma_0-F[2c_1 t]\}. \tag{A 8}$$

Eliminating $\Delta(t)$, $h[(c_0+c_1)\,t]$ in turn to determine the elastic stress distribution and the discontinuity, then evaluating the time derivatives at the interface, shows that

$$2c_0\Delta'(t) = (c_0^2-c_1^2)\,(\partial\sigma/\partial t)_{\mathrm{e}}. \tag{A 9}$$

But validity of the elastic stress distribution requires $(\partial\sigma/\partial t)_{\mathrm{e}} \leqslant 0$ which again implies $\Delta(t) \leqslant 0$ for all $t \geqslant 0$, so that no discontinuity forms.

(iii) $X(t) = -c_1 t \quad (t \geqslant 0)$,

$$Y[-c_1 t] = G[-(c_0+c_1)\,t]+h[(c_0-c_1)\,t] = \sigma_0+e[-2c_1 t]-\Delta(t), \tag{A 10}$$

$$c_1\{G[-(c_0+c_1)\,t]-\sigma_0-h[(c_0-c_1)\,t]\} = c_0\{e[-2c_1 t]+\Delta(t)\}. \tag{A 11}$$

Eliminating $e[-2c_1 t]$ and then $\Delta(t)$, $h[(c_0-c_1)\,t]$ in turn leads to

$$2c_0^2\Delta'(t) = -(c_0^2-c_1^2)\,(\partial\sigma/\partial t)_{\mathrm{e}}. \tag{A 12}$$

But here validity of the elastic stress distribution requires $(\partial\sigma/\partial t)_{\mathrm{e}} \geqslant 0$ so again $\Delta(t) \leqslant 0$ for all $t \geqslant 0$ and no discontinuity forms.

(iv) $X(t) = -c_0 t \quad (t \geqslant 0)$,

$$Y[-c_0 t] = G[-2c_0 t]+\Delta(t) = f[-(c_0-c_1)\,t]+e[-(c_0+c_1)\,t], \tag{A 13}$$

$$c_0\{\sigma_0-f[-(c_0-c_1)\,t]+e[-(c_1+c_1)\,t]\} = c_1\{G[-2c_0 t]-\sigma_0-\Delta(t)\}. \tag{A 14}$$

Eliminating $\Delta(t)$, $f[-(c_0-c_1)\,t]$, $e[-(c_0+c_1)\,t]$ in turn leads to

$$2c_1^2\Delta'(t) = -(c_0^2-c_1^2)\,(\partial\sigma/\partial t)_{\mathrm{p}}, \tag{A 15}$$

so that the plastic loading requirement $(\partial\sigma/\partial t)_{\mathrm{p}} \geqslant 0$ again implies $\Delta(t) \leqslant 0$ for all $t \geqslant 0$ and no discontinuity forms.

Appendix B. Solution for special linear profiles

The conditions derived in §5 for the validity of the different solutions to the initial meeting interaction allow, for the initial linear profiles

$$\left.\begin{aligned} G'(x) &\equiv \lambda \quad (x \leqslant 0) \\ F'(x) &\equiv \lambda \quad (x \geqslant 0) \end{aligned}\right\} \quad (\lambda > 0) \tag{B 1}$$

the type C, B and D solutions. That is, they fail to select a unique type of solution. Solutions of type E and F, which require $2G' \leqslant Y'$, are not possible initially since $Y_0' \leqslant 0$. It is now shown that these three valid solutions for the initial profiles (B 1) give identical stress distributions for some interval $t \geqslant 0$, and their formal difference arises from the lack of an explicit interface. Analogous situations in the general interaction are $2F' \equiv 2G' \equiv Y' \equiv$ constant, $2H' \equiv 2G' \equiv Y' \equiv$ constant, and $E' \equiv F' \equiv G' \equiv$ constant, where the initial distributions apply in their appropriate regions, and may be similarly interpreted.

Type C, $0 < \dot{X} < c_1$.

Substituting the initial profiles defined by (B 1), together with $G(0) = F(0) = \sigma_0$, into (4·1·10), (4·1·11) gives for $t \geqslant 0$,

$$h[X(t)+c_0 t] = \lambda\{X(t)+c_0 t\}, \quad e[X(t)-c_1 t] = \lambda\{X(t)-c_1 t\}. \tag{B 2}$$

That is,

$$h(x) = \lambda x \quad (x \geqslant 0); \quad e(x) = \lambda x \quad (x \leqslant 0); \tag{B 3}$$

independent of $X(t)$. Then by (4·1·2), the stress in the elastic domain PON (figure 5) is

$$\sigma(x,\, t) = \sigma_0 + 2\lambda x, \tag{B 4}$$

and is just the initial elastic wave in the domain AON. By (4·1·3), the stress in the plastic domain POZ is

$$\sigma(x, t) = \sigma_0 + 2\lambda x, \tag{B 5}$$

and in the domain DOZ is just the initial plastic wave. Thus, independent of the interface path $X(t)$, the stress has the steady (time-independent) distribution (B 4) or (B 5) in the domain NOZ. That is, the stress at each particle neither unloads nor loads, and validity in both elastic and plastic regions is just satisfied. In particular, each particle in $x \geqslant 0$, beyond OZ, remains just at yield, and the plastic-elastic interface path is not strictly defined, but could be positioned anywhere in JOZ without contradiction.

Type B, $c_1 \leqslant \dot{X} \leqslant c_0$.

By (5·2·6) it is seen that the initial conditions (B 1) imply $\dot{X} \equiv c_1$, a limit velocity of the range, and then (4·2·2) shows that

$$h(x) = \lambda x \quad (x \geqslant 0); \tag{B 6}$$

here $e(x) \equiv 0$. In figure 6 the interface path OP is now just the limit characteristic OZ of figure 5, and the elastic region is the domain NOZ, once again with stress distribution (B 4) or (B 5). Outside NOZ the stress is given by the respective initial elastic and plastic waves. Thus this solution is identical to the type C solution.

Type D, $-c_1 \leqslant \dot{X} \leqslant 0$.

Since $Y_0' \leqslant 0$, (B 1) implies that for some initial interval $Y' < 2G' \equiv 2F'$ evaluated on the interface path, and hence by (4·3·3) $\dot{X} \equiv 0$, so that the interface path is OJ in figure 5. Substituting in (4·1·10), (4·1·11) again leads to (B 3) and (B 4), (B 5) for the elastic and plastic stress distributions over the domain NOZ, identical to the previous solutions.

Appendix C. Interface acceleration at end points of velocity ranges

In each of the general solutions the interface velocity equation can be expressed in the form

$$[\dot{X}(t)-q]\,\Omega(t) = \pm\alpha^2\Psi(t), \tag{C 1}$$

where q is a limit velocity of the range, α is constant, and the $\pm$ signs apply when q is the lower and upper limit respectively. $\Omega(t)$, $\Psi(t)$ depend on appropriate initial wave and yield stress functions evaluated on the interface path $X(t)$, defined as follows:

$$\left.\begin{aligned} A:\quad &\Omega(t) = F''[X(t)+c_1 t]-E''[X(t)-c_1 t];\\ &q=c_0,\quad \Psi(t) = -(c_0+c_1)\,F''[X(t)+c_1 t]+(c_0-c_1)\,E''[X(t)-c_1 t].\end{aligned}\right\} \tag{C 2}$$

$$\left.\begin{aligned} B:\quad &\Omega(t) = 2c_1 G'[X(t)-c_0 t]+(c_0-c_1)\,F'[X(t)+c_1 t]-(c_0+c_1)\,E'[X(t)-c_1 t];\\ &q=c_0,\quad \Psi(t) = F'[X(t)+c_1 t]-E'[X(t)-c_1 t];\\ &q=c_1,\quad \Psi(t) = G'[X(t)-c_0 t]-F'[X(t)+c_1 t].\end{aligned}\right\} \tag{C 3}$$

$$\left.\begin{aligned} C:\quad &\Omega(t) = F''[X(t)+c_1 t]-G''[X(t)-c_0 t];\\ &q=c_1,\quad \Psi(t) = (c_0-c_1)\,G''[X(t)-c_0 t]+2c_1 F''[X(t)+c_1 t];\\ &q=0,\quad \Psi(t) = -c_1 F''[X(t)+c_1 t]-c_0 G''[X(t)-c_0 t].\end{aligned}\right\} \tag{C 4}$$

$$\left.\begin{aligned} D:\quad &\Omega(t) = 2c_0 F'[X(t)+c_1 t]+2c_1 G'[X(t)-c_0 t]-(c_0+c_1)\,Y'[X(t)];\\ &q=0,\quad \Psi(t) = F'[X(t)+c_1 t]-G'[X(t)-c_0 t];\\ &q=-c_1,\quad \Psi(t) = 2G'[X(t)-c_0 t]-Y'[X(t)].\end{aligned}\right\} \tag{C 5}$$

$$\left.\begin{aligned} E:\quad &\Omega(t) = Y''[X(t)]-2G''[X(t)-c_0 t];\\ &q=-c_1,\quad \Psi(t) = 2(c_0+c_1)\,G''[X(t)-c_0 t]-c_1 Y''[X(t)];\\ &q=-c_0,\quad \Psi(t) = -4G''[X(t)-c_0 t]+Y''[X(t)].\end{aligned}\right\} \tag{C 6}$$

$$\left.\begin{aligned} F:\quad &\Omega(t) = G'[X(t)-c_0 t]+H'[X(t)+c_0 t]-Y'[X(t)];\\ &q=-c_0,\quad \Psi(t) = -2G'[X(t)-c_0 t]+Y'[X(t)].\end{aligned}\right\} \tag{C 7}$$

For simplicity assume that the limit velocity q occurs at $t=0$ with $X(0)=0$. In each case $\dot{X}(0)=q$ implies that $\Psi(0)=0$, given by equality between the two appropriate component function derivatives at $t=0$. The same function derivatives occur in the corresponding $\Omega(0)$, together with a third function derivative for types B, D and F. In these types, if equality does not hold between all three function derivatives, then the respective validity conditions demand

$$\left.\begin{aligned} &B:\quad E'=F'<G' \quad\text{or}\quad E'<F'=G',\\ &D:\quad Y'<2G'=2F' \quad\text{or}\quad Y'=2G'<2F',\\ &F:\quad Y'=2G'<2H',\end{aligned}\right\} \tag{C 8}$$

all evaluated at $t=0$, and in each case it follows that

$$\Omega(0)>0. \tag{C 9}$$

When the equality between all three does hold, and $\Omega(0)=0$, the lowest derivative $\Omega^{(K)}(0)$, $K\geqslant 1$, for $\dot{X}=q$, associated with the lowest non-vanishing derivative of the components in the corresponding $\Psi'(t)$, is independent of the third function since in each case this has argument $X(t)-qt$. Now differentiating (C 1) shows that

$$(\dot{X}-q)\,\Omega^{(K)}(t)+\sum_{s=1}^{K}\binom{K}{s}\dot{X}^{(s)}\,\Omega^{(K-s)}(t)=\pm\alpha^{2}\Psi'^{(K)}(t), \tag{C 10}$$

and since all terms in the series vanish at $t=0$ by definition of K, $\dot{X}=q$ implies that $\Psi'^{(K)}(0)=0$, and hence the lowest non-vanishing derivatives at $t=0$ of both components must be the same order. With this restriction and the consequent validity conditions, evaluated at $t=0$,

$$\left.\begin{array}{ll} B: & F'=E'=G'; \\ & q=c_0,\quad L=M=K+1,\quad E^{(M)}<0\quad (6\cdot1\cdot10),\quad \mathscr{A}\langle M\rangle=\Psi'^{(K)}(0)=0; \\ & q=c_1,\quad M=N=K+1,\quad F^{(M)}>0\quad (6\cdot1\cdot12),\quad \mathscr{B}\langle M\rangle=\Psi'^{(K)}(0)=0;\end{array}\right\} \tag{C 11}$$

$$\left.\begin{array}{ll} D: & 2F'=2G'=Y'; \\ & q=0,\quad M=N=K+1,\quad F^{(M)}>0\quad (5\cdot3\cdot4),\quad \mathscr{C}\langle M\rangle=\Psi'^{(K)}(0)=0; \\ & q=-c_1,\quad S=N=K+1,\quad \mathscr{D}\langle N\rangle=\Psi'^{(K)}(0)=0, \\ & \qquad\Rightarrow(-1)^{N}Y^{(N)}>0\quad (5\cdot3\cdot13);\end{array}\right\} \tag{C 12}$$

$$\left.\begin{array}{ll} F: & 2H'=2G'=Y'; \\ & q=-c_0,\quad S=N=K+1,\quad (-1)^{N}G^{(N)}>0\quad (6\cdot3\cdot13), \\ & \mathscr{E}\langle N\rangle=\Psi'^{(K)}(0)=0;\end{array}\right\} \tag{C 13}$$

it follows that
$$\Omega^{(K)}(0)>0. \tag{C 14}$$

In view of (C 9), (C 14) applies to the first non-vanishing contribution for $K\geqslant 0$.

Similarly, the restricted validity conditions for types A, C and E,

$$A:\quad q=c_0,\quad L=M=K+2,\quad F^{(M)}<0\quad (6\cdot2\cdot5),\quad \mathscr{A}\langle M\rangle=\Psi'^{(K)}(0)=0; \tag{C 15}$$

$$\left.\begin{array}{ll} C: & M=N=K+2,\quad F^{(M)}>0\quad (5\cdot1\cdot14); \\ & q=c_1,\quad \mathscr{B}\langle M\rangle=\Psi'^{(K)}(0)=0;\quad q=0,\quad \mathscr{C}\langle M\rangle=\Psi'^{(K)}(0)=0;\end{array}\right\} \tag{C 16}$$

$$\left.\begin{array}{ll} E: & S=N=K+2,\quad (-1)^{N}G^{(N)}>0\quad (5\cdot4\cdot6); \\ & q=-c_1,\quad \mathscr{D}\langle N\rangle=\Psi'^{(K)}(0)=0;\quad q=-c_0,\quad \mathscr{E}\langle N\rangle=\Psi'^{(K)}(0)=0;\end{array}\right\} \tag{C 17}$$

again lead to (C 14) for $K\geqslant 0$. Thus, in all cases, (C 10) and (C 14) show that

$$\dot{X}=q\Leftrightarrow\Psi'^{(K)}(0)=0, \tag{C 18}$$

where $\Psi'^{(K)}(0)$ is the appropriate one of $\mathscr{A}\langle M\rangle$ to $\mathscr{E}\langle N\rangle$.

Continued differentiation of (C 10) gives for $r\geqslant 1$,

$$(\dot{X}-q)\,\Omega^{(K+r)}(t)+\sum_{s=1}^{K+r}\binom{K+r}{s}\dot{X}^{(s)}\Omega^{(K+r-s)}(t)=\pm\alpha^{2}\,\Psi'^{(K+r)}(t). \tag{C 19}$$

Now the first term, and the terms for $s=r+1,\ldots,K+r$ in the series all vanish at $t=0$, and the coefficient of $\dot{X}^{(r)}$ is $\binom{K+r}{s}\Omega^{(K)}(0)$, >0. Thus applying (C 19) for $r=1,2,\ldots$ in turn shows that

$$\dot{X}^{(r)}=0\Leftrightarrow\Psi'^{(K+r)}(0)=0\quad (r=1,\ldots,m-1), \tag{C 20}$$

if $\Psi^{(K+m)}(0)$, $m \geqslant 1$, is the lowest non-vanishing derivative. Further, setting $r = m$,

$$\binom{K+m}{m}\Omega^{(K)}(0)\,\dot{X}^{(m)} = \pm\alpha^2\Psi^{(K+m)}(0). \tag{C 21}$$

In terms of new variables $y(t)$, $z(t)$ defined by

$$A \text{ and } B, \quad q = c_0, \quad y = \frac{X+c_1 t}{c_0+c_1}, \quad z = \frac{X-c_1 t}{c_0-c_1}; \tag{C 22}$$

$$B \text{ and } C, \quad q = c_1, \quad y = \frac{c_0 t-X}{c_0-c_1}, \quad z = \frac{X+c_1 t}{2c_1}; \tag{C 23}$$

$$C \text{ and } D, \quad q = 0, \quad y = \frac{X+c_1 t}{c_1}, \quad z = \frac{c_0 t-X}{c_0}; \tag{C 24}$$

$$D \text{ and } E, \quad q = -c_1, \quad y = \frac{c_0 t-X}{c_0+c_1}, \quad z = -\frac{X}{c_1}; \tag{C 25}$$

$$E \text{ and } F, \quad q = -c_0, \quad y = \frac{c_0 t-X}{2c_0}, \quad z = -\frac{X}{c_0}; \tag{C 26}$$

it follows that
$$\Psi(t) = \psi_1(y) - \psi_2(z), \tag{C 27}$$

where $\psi_1(y)$, $\psi_2(z)$ are the respective function derivatives given in order by (C 2) to (C 7). At $t = 0$ ($\dot{X} = q$),

$$y = z = 0, \quad \dot{y} = \dot{z} = 1, \quad \dot{y}^{(r)} = \dot{z}^{(r)} = 0 \quad (r = 1, \ldots, m-1), \tag{C 28}$$

using (C 20), and

$$\Psi^{(j)}(0) = \psi_1^{(j)}(0) = \psi_2^{(j)}(0) = 0 \quad (j = 1, \ldots, K-1), \tag{C 29}$$

where $^{()}$ denotes derivative with respect to argument.

Now $\psi_1^{(K+r)}(0)$, $\psi_2^{(K+r)}(0)$, $r \geqslant 0$, are the respective component terms of $\mathscr{A}\langle M+r\rangle$ to $\mathscr{E}\langle N+r\rangle$; for example:

$$B: \quad \psi_1^{(K+r)}(0) - \psi_2^{(K+r)}(0) = \begin{Bmatrix} -\mathscr{A}\langle M+r\rangle, & q = c_0; \\ \mathscr{B}\langle M+r\rangle, & q = c_1. \end{Bmatrix} \tag{C 30}$$

In particular, for $r = 0$,
$$\Psi^{(K)}(0) = \psi_1^{(K)}(0) - \psi_2^{(K)}(0) = 0, \tag{C 31}$$

but it remains to shows that

$$\left.\begin{aligned} \Psi^{(K+r)}(0) = 0 &\Leftrightarrow \psi_1^{(K+r)}(0) - \psi_2^{(K+r)}(0) = 0 \quad (r = 1, \ldots, m-1), \\ \Psi^{(K+m)}(0) > 0 &\Leftrightarrow \psi_1^{(K+m)}(0) - \psi_2^{(K+m)}(0) > 0, \end{aligned}\right\} \tag{C 32}$$

to verify the text expressions of the interface acceleration results (C 20), (C 21). One further consequence of the validity conditions (C 11) to (C 13), (C 15) to (C 17), is that in all cases

$$\psi_1^{(K)}(0)\,[\dot{y}-\dot{z}]^{(m)}(0) \propto \mp\dot{X}^{(m)}(0), \tag{C 33}$$

where the $\mp$ signs apply when q is the lower and upper limit respectively.

Now following Goodstein (1948), p. 91, and using (C 29), it follows that

$$[\psi(y)]^{(K+r)}(t) = \sum_{j=K}^{K+r} P_j^{K+r}\{y\}\,\psi^{(j)}(y) \quad (r \geqslant 1), \tag{C 34}$$

where $P_j^{K+r}\{y\}$ depends only on $y(t)$ and derivatives, independent of the function $\psi(y)$. In particular, at $t = 0$,

$$y = 0, \quad P_j^{K+r}\{0\} = \frac{1}{j!}[y^j]^{(K+r)}, \quad P_{K+r}^{K+r}\{0\} = (\dot{y})^{K+r} = 1. \tag{C 35}$$

But at $y = 0$ we can deduce the general composition

$$[y^j]^{(n)} = \sum_i \gamma_i \prod_{s=1}^{n} [y^{(s)}]^{\nu_s^i}, \tag{C 36}$$

where the γ_i are non-negative constants and the ν_s^i are non-negative integers. Distinct products are implied for each i. Making the linear transformations $t = \lambda t^*$, $y = \mu y^*$, and comparing the two forms, shows that for each i,

$$\sum_{s=1}^{n} \nu_s^i = j, \quad \sum_{s=1}^{n} s\nu_s^i = n; \quad \sum_{s=2}^{n} (s-1)\,\nu_s^i = n-j. \tag{C 37}$$

If $n = j$ the only solution $(i = 1)$ is $\nu_1^1 = n$, $\nu_s^1 = 0\ (s \geqslant 2)$, which leads to the last result of (C 35). For $n-j = k-1$ $(k \geqslant 2)$, the maximum possible s for which $\nu_s^i \neq 0$, that is the highest derivative of y occurring, is $s = k$. Immediately $P_j^{K+r}\{y\}$ at $y = 0$ involves only derivatives up to $\dot{y}^{(K+r-j)}$, and hence for each $j = K, \ldots, K+r$ and each $r \leqslant m-1$, when $K+r-j \leqslant m-1$ in view of (C 28), it has a single positive term in $\dot{y}$, γ_j^r say. Further, the same result applies to $P_j^{K+r}\{z\}$ at $z = 0$, by (C 28), so that

$$\Psi^{(K+r)}(0) = \sum_{j=K}^{K+r} \gamma_j^r [\psi_1^{(j)}(0) - \psi_2^{(j)}(0)] \quad (r = 1, \ldots, m-1), \tag{C 38}$$

which applied to $r = 1, \ldots, m-1$ in turn and using (C 31) verifies the first set of implications in (C 32).

Finally, setting $r = m$ in (C 34),

$$[\Psi(y)]^{(K+m)}(t) = P_{K+m}^{K+m}\{y\}\,\psi^{(K+m)}(y) + \sum_{j=K+1}^{K+m-1} P_j^{K+m-1}\{y\}\,\psi^{(j)}(y) + P_K^{K+m}\{y\}\,\psi^{(K)}(y), \tag{C 39}$$

where, repeating the above arguments and using (C 38), each term in the series is the same when applied to $\psi_1(y)$ and $\psi_2(z)$ and evaluated at $y = z = 0$. Thus

$$\Psi^{(K+m)}(0) = \psi_1^{(K+m)}(0) - \psi_2^{(K+m)}(0) + \psi_1^{(K)}(0)\,[P_K^{K+m}\{y\} - P_K^{K+m}\{z\}], \tag{C 40}$$

where the last bracket is evaluated at $y = z = 0$. But at $y = 0$, $P_K^{K+m}\{y\}$ has the form $\gamma(\dot{y})^{\nu_1}(\dot{y}^{(m)})^{\nu_{m+1}}$ where $\gamma > 0$ and

$$\nu_1 + \nu_{m+1} = K, \quad m\nu_{m+1} = m. \tag{C 41}$$

That is, $\nu_{m+1} = 1$, $\nu_1 = K-1$, so that $P_K^{K+m}\{y = 0\} = \gamma\dot{y}^{(m)}$, and hence the final term of (C 40) becomes

$$\gamma\psi_1^{(K)}(0)\,[\dot{y} - \dot{z}]^{(m)}(0). \tag{C 42}$$

Combining (C 21), (C 33), (C 40) and (C 42), shows that

$$\dot{X}^{(m)} \propto \pm[\psi_1^{(K+m)}(0) - \psi_2^{(K+m)}(0)], \tag{C 43}$$

or equivalently the last implication of (C 32).

Appendix D. Existence and uniqueness of solutions of the interface path equations

In each of the six types of general solution, the interface path $x = X(t)$ is determined by an implicit equation of the form

$$\phi(x, t) = 0, \tag{D 1}$$

where

$$A: \quad (6\cdot2\cdot1) \quad \phi = F'(x+c_1 t) - E'(x-c_1 t); \tag{D 2}$$

$$B: \quad (6\cdot1\cdot1) \quad \phi = (c_0-c_1)\{F(x+c_1 t)-Y_0\}+2c_1\{G(x-c_0 t)-Y_0\}-(c_0+c_1)\,E(x-c_1 t); \tag{D 3}$$

$$C: \quad (4\cdot1\cdot13) \quad \phi = F'(x+c_1 t) - G'(x-c_0 t); \tag{D 4}$$

$$D: \quad (4\cdot3\cdot2) \quad \phi = 2c_0\{F(x+c_1 t)-Y_0\}+2c_1\{G(x-c_0 t)-Y_0\}-(c_0+c_1)\{Y(x)-Y_0\}; \tag{D 5}$$

$$E: \quad (4\cdot4\cdot4) \quad \phi = Y'(x) - 2G'(x-c_0 t); \tag{D 6}$$

$$F: \quad (6\cdot3\cdot1) \quad \phi = G(x-c_0 t) + H(x+c_0 t) - Y(x). \tag{D 7}$$

Here it is assumed that the path starts at (0, 0), so that $\phi(0, 0) = 0$. Then, provided that $\partial\phi/\partial x$, $\partial\phi/\partial t$ are continuous in a neighbourhood of (0, 0) and either $\partial\phi/\partial x$ or $\partial\phi/\partial t \neq 0$ at (0, 0), and hence in some neighbourhood, there exists a unique solution $x = X(t)$ in that neighbourhood which passes through (0, 0). See, for example, Burkill (1962), p. 167. The previous restrictions ensure that the interface velocity $\dot{X}(t)$ is within the required range.

It remains to examine the situations when both $\partial\phi/\partial x$ and $\partial\phi/\partial t = 0$ at (0, 0). In each type we are concerned only with solutions in a restricted domain, bounded by limit lines $x = q_1 t$ and $x = q_2 t$ where q_1, q_2 are the appropriate limit velocities for the given type. Figure 13 shows a typical domain bounded by limit lines OQV, OWR. For the existence of

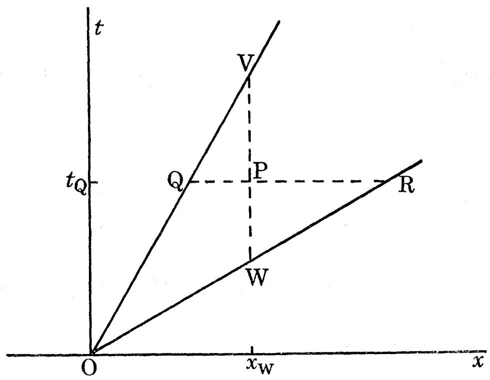

Figure 13. Typical domain for an interface path equation.

a solution within such a domain it is sufficient to show that $\phi < 0$ (say) on $x = q_1 t$, $\phi > 0$ on $x = q_2 t$ ($t > 0$), then for each fixed $t_Q > 0$, ϕ must vanish somewhere on QR, and for each fixed $x_W \neq 0$, ϕ must vanish somewhere on WV. If, further, ϕ is strictly monotonic along QR or WV, then there is a single point at which ϕ vanishes, and the solution is unique. The former property will now be verified for all the remaining cases, ensuring existence. Also the latter property is verified except for a few particular sets of initial function conditions, each of which is one of the sets, marked * in the tables, not sufficient to ensure a valid stress wave solution of the given type. In these few cases the present uniqueness proof for the interface path fails, possibly because stronger validity conditions are required and, or, possibly because the monotonicity condition is too strong.

Following are the conditions for each type when both $\partial\phi/\partial x$ and $\partial\phi/\partial t$ vanish at (0, 0), together with expressions for the first non-vanishing $\phi^{(m)} = \mathrm{d}^m\phi/\mathrm{d}t^m$ along $x = q_1 t$ and $x = q_2 t$, evaluated at (0, 0), and the text references to restrictions which verify the given signs and hence $\phi \lessgtr 0$ respectively over some finite section adjacent to (0, 0). Along the limit line $t = 0$, $x > 0$, of type A, $\phi^{(m)} = \mathrm{d}^m\phi/\mathrm{d}x^m$.

$$A:\quad \left.\begin{array}{l} F'' = E'' = 0, \quad L = M; \\ t = 0, \quad \phi^{(m)} = F^{(m+1)} - E^{(m+1)} > 0 \quad (6\cdot2\cdot6), \\ q_1 = c_0, \quad \phi^{(m)} = (c_0+c_1)^m F^{(m+1)} - (c_0-c_1)^m E^{(m+1)} < 0 \quad (6\cdot2\cdot7). \end{array}\right\} \tag{D 8}$$

$$B:\quad \left.\begin{array}{l} G' = F' = E'; \\ q_2 = c_0, \quad \phi^{(m)} = (c_0^2-c_1^2)\{(c_0+c_1)^{m-1} F^{(m)} - (c_0-c_1)^{m-1} E^{(m)}\} > 0, \\ \qquad L < M \quad (6\cdot1\cdot10), \quad L > M \quad (6\cdot1\cdot12), \quad L = M \quad (6\cdot1\cdot15); \\ q_1 = c_1, \quad \phi^{(m)} = 2c_1(c_0-c_1)\{(2c_1)^{m-1} F^{(m)} - (c_0-c_1)^{m-1}(-1)^{m-1} G^{(m)}\} < 0, \\ \qquad N \neq M \quad (6\cdot1\cdot18), \quad N = M \quad (6\cdot1\cdot19). \end{array}\right\} \tag{D 9}$$

$$C:\quad \left.\begin{array}{l} F'' = G'' = 0, \quad M = N; \\ q_2 = c_1, \quad \phi^{(m)} = (2c_1)^m F^{(m+1)} - (c_0-c_1)^m (-1)^m G^{(m+1)} > 0 \quad (5\cdot1\cdot12); \\ q_1 = 0, \quad \phi^{(m)} = c_1^m F^{(m+1)} - c_0^m (-1)^m G^{(m+1)} < 0 \quad (5\cdot1\cdot12). \end{array}\right\} \tag{D 10}$$

$$D:\quad \left.\begin{array}{l} 2F' = 2G' = Y'; \\ q_2 = 0, \quad \phi^{(m)} = 2c_1 c_0\{c_1^{m-1} F^{(m)} - c_0^{m-1}(-1)^{m-1} G^{(m)}\} > 0; \\ \qquad M < N \quad (5\cdot3\cdot4), \quad M > N \quad (5\cdot3\cdot6), \quad M = N \quad (5\cdot3\cdot8); \\ q_1 = -c_1, \quad \phi^{(m)} = c_1(c_0+c_1)\{2(c_0+c_1)^{m-1}(-1)^m G^{(m)} - c_1^{m-1}(-1)^m Y^m\} < 0, \\ \qquad S \neq N \quad (5\cdot3\cdot12), \quad S = N \quad (5\cdot3\cdot17). \end{array}\right\} \tag{D 11}$$

$$E:\quad \left.\begin{array}{l} Y'' = 2G'' = 0, \quad S = N; \\ q_2 = -c_1, \quad \phi^{(m)} = c_1^m(-1)^m Y^{(m+1)} - 2(c_0+c_1)^m(-1)^m G^{(m+1)} > 0 \quad (5\cdot4\cdot12); \\ q_1 = -c_0, \quad \phi^{(m)} = c_0^m\{(-1)^m Y^{(m+1)} - 2^{m+1}(-1)^m G^{(m+1)}\} < 0 \quad (5\cdot4\cdot13). \end{array}\right\} \tag{D 12}$$

$$F:\quad \left.\begin{array}{l} 2G' = 2H' = Y'; \\ q_2 = -c_0, \quad \phi^{(m)} = c_0^m\{2^m(-1)^m G^{(m)} - (-1)^m Y^{(m)}\} > 0, \\ \qquad S < N \quad (5\cdot5\cdot13), \quad S > N \quad (6\cdot3\cdot13), \quad S = N \quad (5\cdot5\cdot15); \\ t = 0, \quad \phi = -\{Y(x) - \sigma(x, 0)\} < 0 \quad (\text{elastic}). \end{array}\right\} \tag{D 13}$$

Finally, the monotonicity condition at a general point $P(\alpha t, t)$, $q_1 < \alpha < q_2$, is checked by examining the leading term in the expansion for $\partial\phi/\partial x$ or $\partial\phi/\partial t$ about (0, 0), $t \ll 1$, denoted respectively by ϕ_x, ϕ_t after removal of the factor $t^m/m!$.

$$A:\quad \left.\begin{array}{l} \alpha > c_0, \quad L = M; \\ \phi_t = c_1\{(c_1+\alpha)^m F^{(m+2)} + (\alpha-c_1)^m E^{(m+2)}\} < 0 \quad (6\cdot2\cdot5). \end{array}\right\} \tag{D 14}$$

$$\left.\begin{aligned}
&B:\quad c_1<\alpha<c_0;\\
&\phi_x=(c_0-c_1)\,(c_1+\alpha)^m F^{(m+1)}+2c_1(c_0-\alpha)^m\,(-1)^m\,G^{(m+1)}\\
&\qquad\qquad -(c_0+c_1)\,(\alpha-c_1)^m E^{(m)}>0,\\
&\qquad L<M,\quad N\quad(6\cdot1\cdot10),\quad M<N\quad(6\cdot1\cdot12),\quad M>N\quad(5\cdot1\cdot18),\\
&M=N\leqslant L,\quad F^{(M)}>0\Rightarrow(-1)^{M-1}G^{(M)}>0\quad(6\cdot1\cdot19);\\
&\phi_t=c_1\{(c_0-c_1)\,(c_1+\alpha)^m F^{(m+1)}-2c_1(c_0-\alpha)^m\,(-1)^m\,G^{(m+1)}\\
&\qquad\qquad +(c_0+c_1)\,(\alpha-c_1)^m E^{(m)}\}<0,\\
&M=N,\quad F^{(M)}<0,\quad(-1)^{M-1}G^{(M)}>0,\quad\Rightarrow M=L,\quad \dot{X}\neq c_1\quad(6\cdot1\cdot12).
\end{aligned}\right\}\quad (\mathrm{D}\ 15)$$

$$\left.\begin{aligned}
&C:\quad 0<\alpha<c_1,\quad M=N;\\
&\phi_x=(c_1+\alpha)^m F^{(m+2)}+(c_0-\alpha)^m\,(-1)^{m+1}\,G^{(m+2)}>0\quad(5\cdot1\cdot10).
\end{aligned}\right\}\quad (\mathrm{D}\ 16)$$

$$\left.\begin{aligned}
&D:\quad -c_1<\alpha<0;\\
&\phi_t=2c_0c_1\{(c_1+\alpha)^m F^{(m+1)}+(c_0-\alpha)^m\,(-1)^{m+1}\,G^{(m+1)}\}>0,\\
&\qquad M<N\quad(5\cdot3\cdot4),\quad M>N\quad(5\cdot3\cdot6),\quad M=N,\quad(-1)^N G^{(N)}>0;\\
&\phi_x=2c_0(c_1+\alpha)^m F^{(m+1)}+2c_1(c_0-\alpha)^m\,(-1)^m\,G^{(m+1)}\\
&\qquad\qquad -(c_0+c_1)\,(-\alpha)^m\,(-1)^m\,Y^{(m+1)}>0,\\
&M=N,\quad(-1)^{N-1}G^{(N)}>0\quad\text{and}\quad S\neq N\quad(5\cdot3\cdot12),\\
&\qquad\text{or}\quad S=N,\quad(-1)^N Y^{(N)}>0.
\end{aligned}\right\}\quad (\mathrm{D}\ 17)$$

$$\left.\begin{aligned}
&E:\quad -c_0<\alpha<-c_1,\quad S=N;\\
&\phi_t=2c_0(c_0-\alpha)^m\,(-1)^m\,G^{(m+2)}>0\quad(5\cdot4\cdot6).
\end{aligned}\right\}\quad (\mathrm{D}\ 18)$$

$$\left.\begin{aligned}
&F:\quad \alpha<-c_0;\\
&\phi_t=(c_0-\alpha)^m\,(-1)^{m+1}\,G^{(m+1)}+(-c_0-\alpha)^m\,(-1)^m\,H^{(m+1)}>0,\\
&N<P\quad(6\cdot3\cdot13),\quad N>P,\quad \dot{X}\neq -c_0\quad(6\cdot3\cdot15),\\
&N=P,\quad(-1)^{N-1}H^{(N)}>0.
\end{aligned}\right\}\quad (\mathrm{D}\ 19)$$

The following initial conditions are not sufficient to verify the monotonicity requirement.

$$\left.\begin{aligned}
&B:\quad L=M=N,\quad \dot{X}\neq c_1,\quad F^{(M)}<0,\quad(-1)^{M-1}G^{(M)}<0.\\
&D:\quad M=N=S,\quad(-1)^{N-1}G^{(N)}>0,\quad(-1)^{N-1}Y^{(N)}>0.\\
&F:\quad N>P,\quad \dot{X}=c_0,\quad\text{or}\quad N=P\quad\text{and}\quad(-1)^P H^{(P)}>0.
\end{aligned}\right\}\quad (\mathrm{D}\ 20)$$